Exercises for Module 9: Linear Algebra III

1. Let U, V, W be inner product spaces and $S, T \in \mathcal{L}(U, V)$ and $R \in \mathcal{L}(V, W)$. Show that the following holds
   1. $(S + \alpha T)^* = S^* + \overline{\alpha} T^*$ for all $\alpha \in \mathbb{F}$
   2. $(S^*)^* = S$
   3. $(RS)^* = S^* R^*$
   4. $I^* = I$, where $I \colon U \to U$ is the identity operator on U

1. Let $u \in U, v \in V$.

$$\begin{aligned}
\langle u, (S+\alpha T)^* v\rangle &= \langle (S+\alpha T)u, v\rangle && \text{by def'n of adjoint} \\
&= \langle Su + \alpha Tu, v\rangle && \text{by def of linear map} \\
&= \langle Su, v\rangle + \alpha\langle Tu, v\rangle && \text{by lin. of 1st arg.} \\
&= \langle u, S^* v\rangle + \alpha\langle u, T^* v\rangle \\
&= \langle u, S^* v\rangle + \langle u, \overline{\alpha} T^* v\rangle && \text{by linearity + conjugate symmetry} \\
&= \langle u, (S^* + \overline{\alpha} T^*) v\rangle
\end{aligned}$$

$\therefore (S+\alpha T)^* = S^* + \overline{\alpha} T^*$ by exercise 4

2. Let $u \in U, v \in V$.

$$\begin{aligned}
\langle u, (S^*)^* v\rangle &= \langle S^* u, v\rangle \\
&= \overline{\langle v, S^* u\rangle} && \text{by conjugate symmetry} \\
&= \overline{\langle Sv, u\rangle} && \text{by def of adjoint} \\
&= \langle u, Sv\rangle && \text{by conjugate symmetry}
\end{aligned}$$

$\therefore (S^*)^* = S$ by exercise 4

3. Let $u \in U, w \in W$.

$$\begin{aligned}
\langle u, (RS)^* w\rangle &= \langle RSu, w\rangle \\
&= \langle Su, R^* w\rangle \\
&= \langle u, S^* R^* w\rangle
\end{aligned}$$

$\therefore (RS)^* = S^* R^*$

4. Let $u_1, u_2 \in U$.

Then
$$\begin{aligned}
\langle u_1, I^* u_2\rangle &= \langle Iu_1, u_2\rangle \\
&= \langle u_1, u_2\rangle \\
&= \langle u_1, Iu_2\rangle
\end{aligned}$$

$\therefore I = I^*$

2. Let V be an inner product space and $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be an orthonormal basis and $\mathbf{y} \in V$. Then, $\mathbf{x}$ has a unique representation $\mathbf{y} = \sum_{i=1}^{n} \alpha_i \mathbf{x}_i$. Show that $\alpha_i = \langle \mathbf{y}, \mathbf{x}_i \rangle$ for all $i = 1, \ldots, n$.

$$\begin{aligned}\langle y, x_i \rangle &= \left\langle \sum_{j=1}^{n} \alpha_j x_j, x_i \right\rangle \\ &= \sum_{j=1}^{n} \alpha_j \langle x_j, x_i \rangle \quad \text{by linearity in } 1^{st} \text{ argument} \\ &= \alpha_i \quad \text{since } \langle x_j, x_i \rangle = 1 \text{ if } i=j \text{ \& } 0 \text{ otherwise}\end{aligned}$$

3. Let V be an inner product space and $U \subseteq V$ a subset. Show that $U^{\perp}$ is a subspace of V.

$$U^{\perp} := \{ x \in V : \langle x, u \rangle = 0 \quad \forall u \in U \}$$

We must show $U^{\perp}$ is a subspace of V.

First of all, $0 \in U^{\perp}$ since $\langle 0, u \rangle = 0 \ \forall u \in U$.

Let $x, y \in U^{\perp}$. Then $\langle x+y, u \rangle = \langle x, u \rangle + \langle y, u \rangle$ by linearity in 1^{st} argument
$= 0$

so $x+y \in U^{\perp}$.

Also, if $\alpha \in \mathbb{F}$, $x \in U^{\perp}$, then $\langle \alpha x, u \rangle = \alpha \langle x, u \rangle = 0$, so $\alpha x \in U^{\perp}$.

$\therefore U^{\perp} \subseteq V$ is a subspace.

4. Let $A, B \in M_n(\mathbb{F})$ be similar matrices. Show that their characteristic polynomials coincide.

Proof Let A & B be similar. Then there exists an invertible matrix S such that $A = SBS^{-1}$.

Note that similar matrices have the same determinant:

$$\det(A) = \det(SBS^{-1}) = \det(S)\det(B)\det(S^{-1}) = \det(B)$$

since $\det(S^{-1}) = \det(S)^{-1}$ as $SS^{-1} = I$.

We can write

$$\begin{aligned} A - \lambda I &= SBS^{-1} - \lambda SIS^{-1} \\ &= S(BS^{-1} - \lambda IS^{-1}) \\ &= S(B - \lambda I)S^{-1}. \end{aligned}$$

Therefore if A & B are similar, then $A - \lambda I$ & $B - \lambda I$ are similar, and therefore $\det(A - \lambda I) = \det(B - \lambda I)$. So A, B have the same char. poly.

5. Show that $A \in M_n(\mathbb{C})$ is invertible if and only if $0 \notin \sigma(A)$.

Recall that $\lambda \in \sigma(A)$ means λ is an eigenvalue for A, i.e. $\det(A - \lambda I) = 0$.

$(\Rightarrow)$ By contrapositive.

Suppose $0 \in \sigma(A)$. Then $\det(A - 0I) = 0$.

$\Rightarrow \det(A) = 0$

$\Rightarrow$ A is not invertible by theorem from class.

$(\Leftarrow)$ By contrapositive.

Suppose A is not invertible.

Then $\det(A) = 0$.

$\Rightarrow \det(A - 0I) = 0$

$\Rightarrow 0 \in \sigma(A)$

6. Suppose N is a nilpotent matrix. Show that $\sigma(N) = \{0\}$.

Suppose N is nilpotent. This means $\exists\, k \geq 1$ s.t. $N^k = 0$.

First, we show $\{0\} \subseteq \sigma(N)$.

Since N is nilpotent, $N^k = 0 \Rightarrow \det(N^k) = 0 \Rightarrow \det(N)^k = 0 \Rightarrow \det(N) = 0$.

Thus $0 \in \sigma(N)$ by previous exercise.

To show $\sigma(N) \subseteq \{0\}$, first note that if $v \neq 0$ is an eigenvector associated with λ, then $N^k v = \lambda^k v$.

(By induction: $Nv = \lambda v$ by def of eigenvector, if $N^m v = \lambda^m v$ then $N^{m+1} v = N N^m v = N \lambda^m v = \lambda^m N v = \lambda^m \lambda v = \lambda^{m+1} v$)

Then $N^k v = \lambda^k v \Rightarrow 0 = \lambda^k v \Rightarrow \lambda = 0$ since $v \neq 0$.

Thus if λ is an eigenvalue of N, $\lambda = 0$, so $\sigma(N) \subseteq \{0\}$.

$\therefore \sigma(N) = \{0\}$

7. Let $A \in M_n(\mathbb{C})$ be an invertible matrix. Show that λ is an eigenvalue of A if and only if λ^{-1} is an eigenvalue of A^{-1}.

Let λ be an eigenvalue of A. $\lambda \neq 0$ by exercise 2.

$\Leftrightarrow Av = \lambda v$ where $v \neq 0$ by definition

$\Leftrightarrow A^{-1}Av = A^{-1}\lambda v$

$\Leftrightarrow Iv = \lambda A^{-1} v$

$\Leftrightarrow v = \lambda A^{-1} v$

$\Leftrightarrow \lambda^{-1} v = A^{-1} v$

$\Leftrightarrow \lambda^{-1}$ is an eigenvalue of A^{-1} by definition

8. Suppose $A \in M_n(\mathbb{C})$ is Hermitian. Show that all the eigenvalues of A are real. Hint: Note that if $\mathbf{x}$ is a normalized eigenvector of A with eigenvalue λ, then $\langle A\mathbf{x}, \mathbf{x}\rangle = \lambda$.

Suppose A is Hermitian. This means $A = A^*$.

Let λ be an eigenvalue of A. Then $\exists v \neq 0$ s.t $Av = \lambda v$.

We can normalize v by dividing by $\|v\| = \sqrt{\langle v, v\rangle}$, so

$\exists x \neq 0$ s.t $Ax = \lambda x$ & $\|x\| = 1$.

Then
$$
\begin{aligned}
\lambda = \lambda \|x\|^2 &= \lambda \langle x, x\rangle \\
&= \langle \lambda x, x\rangle && \text{by linearity of 1st argument of inner prod} \\
&= \langle Ax, x\rangle \\
&= \langle x, A^* x\rangle && \text{since } A^* \text{ is the adjoint} \\
&= \langle x, Ax\rangle && \text{since } A = A^* \\
&= \langle x, \lambda x\rangle \\
&= \bar{\lambda} \langle x, x\rangle && \text{by conjugate symmetry \& linearity of inner product} \\
&= \bar{\lambda}
\end{aligned}
$$

Since $\lambda = \bar{\lambda}$, $\lambda \in \mathbb{R}$.

9. Let $A \in M_n(\mathbb{R})$. Show that the eigenvalues of $A^T A$ are non-negative.

Let $A \in M_n(\mathbb{R})$. Note that this means the adjoint of A is A^T.

Let λ be an eigenvalue of A^TA with normalized eigenvector x, i.e. $A^TAx = \lambda x$ & $\|x\| = 1$.

Then
$$
\begin{aligned}
\lambda = \lambda \|x\|^2 = \lambda \langle x, x\rangle &= \langle \lambda x, x\rangle \\
&= \langle A^TAx, x\rangle \\
&= \langle Ax, Ax\rangle && \text{since } (A^T)^* = A \\
&= \|Ax\|^2 \geq 0 && \text{by properties of norm}
\end{aligned}
$$