Exercises for Module 9: Linear Algebra III

1. Let U, V, W be inner product spaces and $S, T \in \mathcal{L}(U, V)$ and $R \in \mathcal{L}(V, W)$. Show that the following holds
    1. $(S + \alpha T)^* = S^* + \overline{\alpha} T^*$ for all $\alpha \in \mathbb{F}$
    2. $(S^*)^* = S$
    3. $(RS)^* = S^* R^*$
    4. $I^* = I$, where $I \colon U \to U$ is the identity operator on U

2. Let V be an inner product space and $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be an orthonormal basis and $\mathbf{y} \in V$. Then, $\mathbf{x}$ has a unique representation $\mathbf{y} = \sum_{i=1}^{n} \alpha_i \mathbf{x}_i$. Show that $\alpha_i = \langle \mathbf{y}, \mathbf{x}_i \rangle$ for all $i = 1, \ldots, n$.

3. Let V be an inner product space and $U \subseteq V$ a subset. Show that $U^{\perp}$ is a subspace of V.

4. Let $A, B \in M_n(\mathbb{F})$ be similar matrices. Show that their characteristic polynomials coincide.

5. Show that $A \in M_n(\mathbb{C})$ is invertible if and only if $0 \notin \sigma(A)$.

6. Suppose N is a nilpotent matrix. Show that $\sigma(N) = \{0\}$.

7. Let $A \in M_n(\mathbb{C})$ be an invertible matrix. Show that λ is an eigenvalue of A if and only if λ^{-1} is an eigenvalue of A^{-1}.

8. Suppose $A \in M_n(\mathbb{C})$ is Hermitian. Show that all the eigenvalues of A are real. Hint: Note that if $\mathbf{x}$ is a normalized eigenvector of A with eigenvalue λ, then $\langle A\mathbf{x}, \mathbf{x} \rangle = \lambda$.

9. Let $A \in M_n(\mathbb{R})$. Show that the eigenvalues of $A^T A$ are non-negative.