# Module 3: Set theory and metrics

## Operational math bootcamp

Statistical Sciences
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# Outline

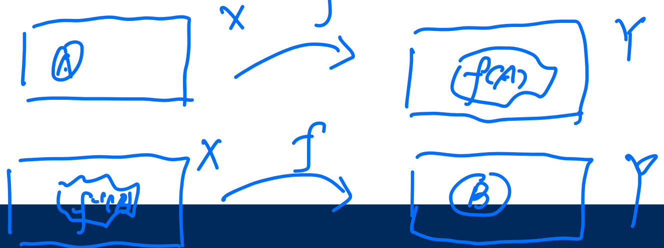
- More on set theory
- Cardinality of sets
- Metrics and norms

# Recall

function $f$ is a subset of $X \times Y$

Image:

pre-image:




## Definition (Image and pre-image)

Let $f: X \to Y$ and $A \subseteq X$ and $B \subseteq Y$.

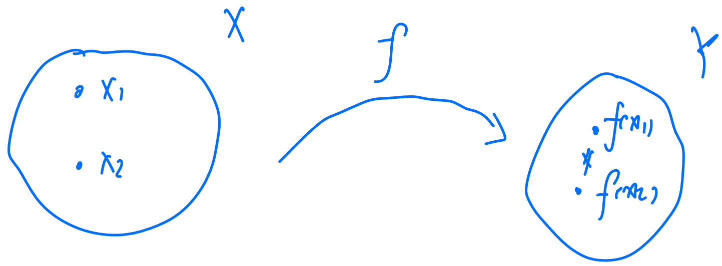
- The *image* of $f$ is the set $f(A) := \{f(x) : x \in A\}$.
- The *pre-image* of $f$ is the set $f^{-1}(B) := \{x : f(x) \in B\}$.

## Definition (Surjective, injective and bijective)

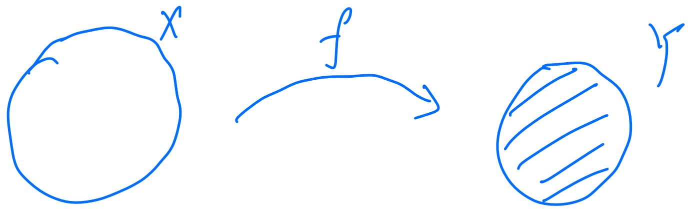
Let $f: X \to Y$, where $X$ and $Y$ are sets. Then

- $f$ is *injective* if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$ Equivalently, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
- $f$ is *surjective* if for every $y \in Y$, there exists an $x \in X$ such that $y = f(x)$
- $f$ is *bijective* if it is both injective and surjective

Injective: $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$




Surjective: $\forall y \in Y$, $\exists x \in X$, s.t. $f(x) = y$




All points are filled in Y.

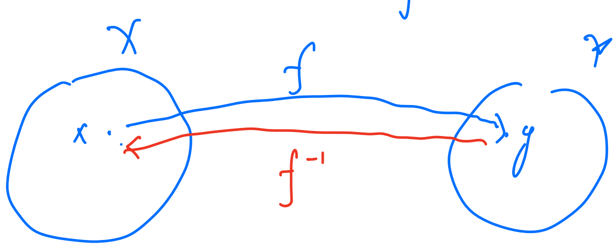
Bijective: Both injective and surjective

$\Leftrightarrow$ $\forall y \in Y$, $\exists! \, x \in X$, s.t. $y = f(x)$

unique

$\Leftrightarrow$ Inverse map $f^{-1}$ is well-defined

or $f^{-1}$ is a well-defined function (it should satisfy the property)



## Proposition

*Let $f : X \to Y$ and $A \subseteq X$. Prove that $A \subseteq f^{-1}(f(A))$, with equality ~~iff~~ if $f$ is injective.*

*Proof.* First, show $A \subseteq f^{-1}(f(A))$ pre-image of the image of $A$

we need to show $\forall a \in A$, we have $a \in f^{-1}(f(A))$, $\iff f(a) \in f(A)$

That's true because $a \in A$

Next we show $\boxed{\text{"if" part.}}$

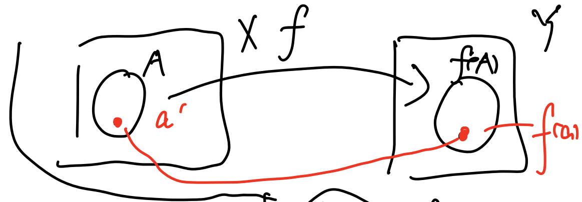
we need to show that, if $f$ is injective,

then $A = f^{-1}(f(A))$

$$\begin{cases} A \subseteq f^{-1}(f(A)) & \text{proved already} \\ f^{-1}(f(A)) \subseteq A \Leftarrow & \end{cases}$$

We need to show $\forall a \in f^{-1}(f(A))$, $\boxed{\text{we } a \in A}$

$\boxed{\forall a \in f^{-1}(f(A))}$, $\Longleftrightarrow$ $f(a) \in f(A)$




$\rightarrow$ Then, $\exists\, a' \in A$ s.t. $f(a) = f(a')$.

since $f$ is injective, $a = a'$ $\quad a' = \boxed{a \in A}$

Thus. $f^{-1}(f(A)) \subseteq A$. and $f^{-1}(f(A)) = A$

$f(a) \in f(A)$

$\exists\, a' \in A$ s.t.

$f(a') = f(a)$.

$\Uparrow$

$\forall x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$

$\Leftrightarrow$ If $f(x_1) = f(x_2)$, then. $x_1 = x_2$

# Cardinality

Intuitively, the *cardinality* of a set $A$, denoted $|A|$, is the number of elements in the set. For sets with only a finite number of elements, this intuition is correct. We call a set with finitely many elements finite.

e.g. $A = \{a, b, c\}$

$|A| = 3$

We say that the empty set has cardinality 0 and is finite.

## Proposition

*If $X$ is finite set of cardinality $n$, then the cardinality of $\mathcal{P}(X)$ is $2^n$.*

*Proof.* consists of "n" elements    power of set, the set of all subset of X

$X = \{x_1, x_2, \cdots, x_n\}$

For $A \subseteq X$, for each $x_i$: $\nearrow \in A$, $\searrow \notin A$

2 options for each i.

since we have "n" element, $i = 1, 2, \cdots, n$.

there are $2^n$ different subsets of X

# Cardinality

Finite sets : number of elements

For infinite sets,

E.g. Are $\mathbb{N}$ and $\aleph_0$ the same size?

$X$ or $Y$ – sets

$x$ or $y$ – elements

△ A and B can be infinite.

## Definition

Two sets $A$ and $B$ have same cardinality, $|A| = |B|$, if there exists bijection $f\colon A \to B$.

## Example

Which is bigger, $\mathbb{N}$ or $\mathbb{N}_0$? show answer: $|\mathbb{N}| = |\mathbb{N}_0|$

$f: \mathbb{N} \to \mathbb{N}_0$, let $f(n) = n-1$

Then we can show $f$ is both injective and surjective

Therefore, by definition of cardinality,

$|\mathbb{N}| = |\mathbb{N}_0|$

$\mathbb{N}$: 1 2 3 4 5 . . .

$f$

$\mathbb{N}_0$: 0 1 2 3 4 5 . . .

$f(x) = 1 \to x = 2$.

Injective: $f: X \to Y$, $\begin{cases} \forall x_1 \neq x_2, & f(x_1) \neq f(x_2) \\ f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \end{cases}$

$f(x) = n \to x = n+1$

Surjective: $\forall y \in Y$, $\exists x \in X$, s.t. $y = f(x)$

$\forall n \in N_0$, $x = n+1$

$f(x) = n+1-1 = n$

# Cantor-Schröder-Bernstein

**Definition**

We say that the cardinality of a set $A$ is less than the cardinality of a set $B$, denoted $|A| \leq |B|$ if there exists an injection $f : A \to B$.

**Theorem (Cantor-Bernstein)**

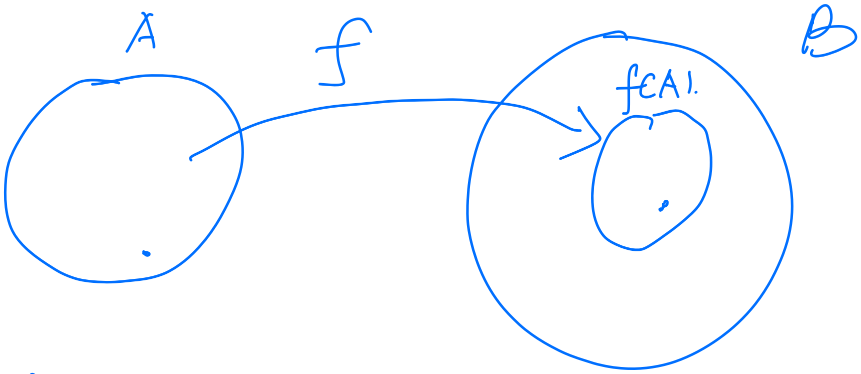
Let $A$, $B$, be sets. If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

By definition, to prove "=", we need find/construct a bijection

**Example**

$|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$

By C-B theorems, it suffices to construction two injection.




a. If $f$ is injective, $f: A \to f(A)$ is bijective

$f: A \to B.$

$$|A| = |f(A)| \quad , \quad f(A) \subseteq B, \quad |f(A)| \le |B|$$

$$|A| \le |B|$$

This doesn't hold for finite set

Proof that $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$: If $|A| = n$, then $|A \times A| = n^2$

By C-B theorem, prove it by two directions.

1) $f: N \to N \times N$, $f(n) = (n, 1)$

Then if $n_1 \neq n_2$, we have $f(n_1) = (n_1, 1) \neq (n_2, 1) = f(n_2)$

Thus $f$ is injective, which means $|\mathbb{N}| \leq |\mathbb{N} \times \mathbb{N}|$

2) $g: N \times N \to \mathbb{N}$, $g(n, m) = 2^n \cdot 3^m$

So $g(n, m) = g(n', m')$, $\Leftrightarrow 2^n 3^m = 2^{n'} 3^{m'} \Leftrightarrow n = n'$ and $m = m'$

Thus $g$ is injective, implies $|\mathbb{N} \times \mathbb{N}| \leq |\mathbb{N}|$

Thus $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$

## Definition

Let $A$ be a set.

1. $A$ is *finite* if there exists an $n \in \mathbb{N}$ and a bijection $f\colon \{1, \dots, n\} \to A$ .
2. $A$ is *countably infinite* if there exists a bijection $f\colon \mathbb{N} \to A$
3. $A$ is *countable* if it is finite or countably infinite $\mathbb{N}\times\mathbb{N}$
4. $A$ is *uncountable* otherwise $\mathbb{R}$ , $(0,1)$

## Example

The rational numbers are countable, and in fact $|\mathbb{Q}| = |\mathbb{N}|$.

*Proof.* First we show $|\mathbb{N}| \leq |\mathbb{Q}^+|$. set of positive rationals

let $f : N \to Q^+$ be $f(n) = n$

Next, we show that $|\mathbb{Q}^+| \leq \underline{|\mathbb{N} \times \mathbb{N}|}. = |\mathbb{N}|$

$\forall\, \varepsilon \in \mathbb{Q}^+ ,\ \exists!\ r, s \in \mathbb{N}$ s.t. $\varepsilon = r/s$ where $r$ and $s$ are mutually prime

↑ Definition of rational number.

Let $g(\varepsilon) = (r, s)$

Then $g : \mathbb{Q}^+ \to \mathbb{N} \times \mathbb{N}$ is injective.

$\left[ \begin{array}{l} \text{let } \varepsilon_1 = r_1/s_1,\ \varepsilon_2 = r_2/s_2, \text{ assume } g(\varepsilon_1) = g(\varepsilon_2) \\ \text{then } (r_1, s_1) = (r_2, s_2) \Leftrightarrow r_1 = r_2,\ s_1 = s_2 \Leftrightarrow \varepsilon_1 = \varepsilon_2 \end{array} \right]$

$|\mathbb{Q}^+| \leq |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$

$|\mathbb{N}| = |\mathbb{Q}^+|$

We can extend this to $\mathbb{Q}$ as follows:

We've already have $|\mathbb{Q}^+| = |\mathbb{N}|$

Next, is to show $|\mathbb{Q}| = |\mathbb{Q}^+|$

## Theorem

The cardinality of $\mathbb{N}$ is strictly smaller than that of $(0, 1)$.

## Proof.

First, we show that there is an injective map from $\mathbb{N}$ to $(0, 1)$.

Let $f: \mathbb{N} \to (0,1)$ $f(n) = \frac{1}{n+1}$ $\Rightarrow |\mathbb{N}| \le |(0,1)|$

Next, we show that there is no surjective map from $\mathbb{N}$ to $(0, 1)$. We use the fact that every number $r \in (0, 1)$ has a binary expansion of the form $r = 0.\sigma_1\sigma_2\sigma_3\ldots$ where $\sigma_i \in \{0, 1\}$, $i \in \mathbb{N}$. □

binary expansion

$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{i=1}^{\infty} (2^{-i})$, $r \in (0,1)$

$r = \sum_{i=1}^{\infty} 2^{-i} \cdot \sigma_i$

E.g. $1 = 0.11111\ldots$ $0.25 = 0.010000\ldots$

$0.5 = 0.10000\ldots$ $0.125 = 0.001000\ldots$

## Proof.

Now we suppose in order to derive a contradiction that there does exist a surjective map $f$ from $\mathbb{N}$ to $(0, 1)$., i.e. for $n \in \mathbb{N}$ we have $f(n) = 0.\sigma_1(n)\sigma_2(n)\sigma_3(n)\ldots$. This means we can list out the binary expansions, for example like

$$\begin{aligned} f(1) &= 0.00000000\ldots \\ f(2) &= 0.1111111111\ldots \\ f(3) &= 0.0101010101\ldots \\ f(4) &= 0.1010101010\ldots \end{aligned}$$

$|\mathbb{N}| < |(0,1)|$

We will construct a number $\tilde{r} \in (0, 1)$ that is not in the image of $f$. $\Rightarrow f$ is not surjective $\square$

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Assume there exists a function from $\mathbb{N}$ to $(0,1)$

$f: \mathbb{N} \to (0,1)$

$$f(1) = 0.\ \sigma_1(1)\ \sigma_2(1)\ \sigma_3(1)\ \sigma_4(1) \cdots$$

$$f(2) = 0.\ \sigma_2(1)\ \sigma_2(2)\ \sigma_3(2)\ \sigma_4(2) \cdots$$

$$f(3) = 0.\ \sigma_3(1)\ \sigma_3(2)\ \sigma_3(3)\ \sigma_4(3) \cdots$$

$$f(4) = 0.\ \sigma_4(1)\ \sigma_4(2)\ \sigma_4(3)\ \sigma_4(4) \cdots$$

$\tilde{r}$ Let $\tilde{\sigma}(1) = \underline{1 - \sigma_1(1)} = \begin{cases} 0, & \text{if } \sigma_1(1) = 1 \\ 1, & \text{if } \sigma_1(1) = 0 \end{cases}$

$\tilde{\sigma}(2) = 1 - \sigma_2(2)$

Let $\tilde{\sigma}(i) = 1 - \sigma_i(i)$

$\tilde{r} = 0.\ \underline{\tilde{\sigma}(1)}\ \tilde{\sigma}(2)\ \tilde{\sigma}(3)\ \tilde{\sigma}(4) \cdots$

Next, we can show $\hat{r}$ is not in the image of $f(\mathbb{N})$

$\forall n \in \mathbb{N}$, $\hat{r}$ and $f(n)$ is different in $n$-th digit

$\therefore$ This holds for any function $: \mathbb{N} \to (0,1)$

**Proof.**

Define $\tilde{r} = 0.\tilde{\sigma}_1\tilde{\sigma}_2\ldots$, where we define the $n$th entry of $\tilde{r}$ to be the the opposite of the $n$th entry of the $n$th item in our list:

$$\tilde{\sigma}_n = \begin{cases} 1 & \text{if } \sigma_n(n) = 0, \\ 0 & \text{if } \sigma_n(n) = 1. \end{cases}$$

Then $\tilde{r}$ differs from $f(n)$ at least in the $n$th digit of its binary expansion for all $n \in \mathbb{N}$. Hence, $\tilde{r} \notin f(\mathbb{N})$, which is a contradiction to $f$ being surjective. This technique is often referred to as Cantor's diagonal argument. $\square$

## Proposition

(0,1) and $\mathbb{R}$ have the same cardinality.

## Proof.


Bijection $f : \mathbb{R} \to (0,1)$

$\Leftarrow$ monotone, cts, $0 < f(x) < 1$

$\lim_{x \to -\infty} f(x) = 0$, $\lim_{x \to +\infty} f(x) = 1$

$\square$

We have shown that there are different sizes of infinity, as the cardinality of $\mathbb{N}$ is infinite but still smaller than that of $\mathbb{R}$ or $(0, 1)$. In fact, we have

$$|\mathbb{N}| = |\mathbb{N}_0| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}|.$$

Because of this, there are special symbols for these two cardinalities: The cardinality of $\mathbb{N}$ is denoted $\aleph_0$, while the cardinality of $\mathbb{R}$ is denoted $\mathfrak{c}$. This is often called as the cardinality of the continuum

In fact there are many other cardinalities, as the following theorem shows:

**Theorem (Cantor's theorem)**

*For any set $A$, $|A| < |\mathcal{P}(A)|$.*

For both finite set and infinite set

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# Metric Spaces

metric is a generalization of "distance"

$\mathbb{R}$, $\quad d(x,y) = |x-y|$

$\mathbb{R}^n$, $\quad$ Euclidean distance

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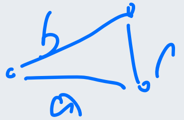
## Definition (Metric)

A *metric* on a set $X$ is a function $d : X \times X \to \mathbb{R}$ that satisfies:

(a) Positive definiteness: $d(x,y) \geq 0$ for $\forall x,y$. and $d(x,y)=0$ iff $x=y$.

(b) Symmetry: $d(x,y) = d(y,x)$

(c) Triangle inequality: $d(x,y) + d(y,z) \geq d(x,z)$




$a+b > c$

A set together with a metric is called a metric space.

$\left\{ \begin{array}{l} \text{set} \\ \text{metric} \end{array} \right.$

## Example ($\mathbb{R}^n$ with the Euclidean distance)

$\forall\, x, y \in \mathbb{R}^n$, $\quad d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$

Norm: size / length of vector

$\begin{cases} \mathbb{F} = \mathbb{R} \\ \mathbb{F} = \mathbb{C} \end{cases}$

$d(\cdot,\cdot) : X \times X \to \mathbb{R}$

## Definition (Norm)

A *norm* on an $\mathbb{F}$-vector space $E$ is a function $\|\cdot\| : E \to \mathbb{R}$ that satisfies:

(a) Positive definiteness: $\|x\| \geq 0, \ \forall x \in E$ and $\|x\| = 0$ iff $x = 0$

(b) Homogeneity: $\forall x \in E, \ \forall \alpha \in \mathbb{F} \ (\mathbb{R}$ or $\mathbb{C})$ $\|\alpha x\| = |\alpha| \cdot \|x\|$

(c) Triangle inequality: $\forall x, y \in E, \ \|x+y\| \leq \|x\| + \|y\|$

A vector space with a norm is called a normed space. A normed space is a metric space using the metric $d(x, y) = \|x - y\|$.

If we have a norm, we can a corresponding metric $d(x,y) = \|x-y\|$

If we have a metric $\not\Rightarrow$ corresponding norm.

## Example ($p$-norm on $\mathbb{R}^n$)

The $p$-norm is defined for $p \geq 1$ for a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ as

$$\|x\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$

when $p = 2$, 2-norm = euclidean norm

$p = 1$, $\|x\|_p = \sum_{i=1}^{n} |x_i|$

The infinity norm is the limit of the $p$-norm as $p \to \infty$, defined as

$$\|x\|_\infty = \max_i |x_i|$$

space of cts functions

## Example ($p$-norm on $C([0,1];\mathbb{R})$ )

If we look at the space of continuous functions $C([0,1];\mathbb{R})$, the $p$-norm is

$$\|f\|_p = \left( \int_0^1 |f(x)|^p dx \right)^{1/p}$$

and the $\infty$−norm (or sup norm) is

$$\|f\|_\infty = \max_{x \in [0,1]} |f(x)|$$

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## Definition

A subset $A$ of a metric space $(X, d)$ is *bounded* if there exists $M > 0$ such that $d(x, y) < M$ for all $x, y \in A$.

## Definition

Let $(X, d)$ be a metric space. We define the *open ball* centred at a point $x_0 \in X$ of radius $r > 0$ as

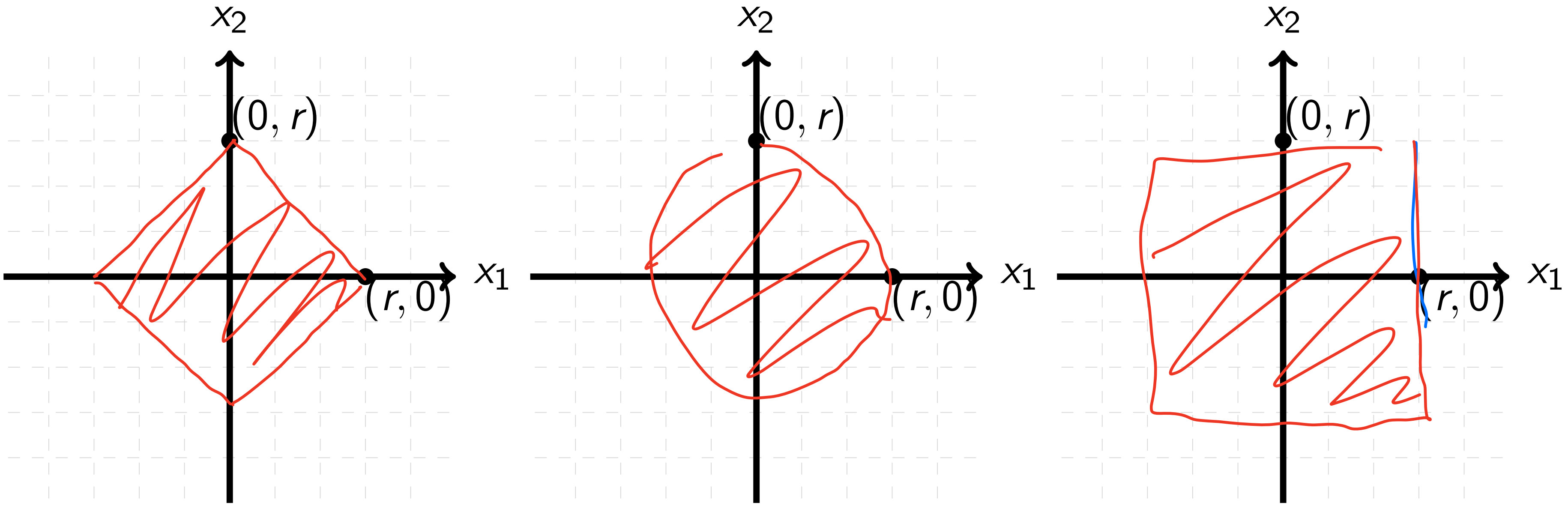
$$B_r(x_0) := \{x \in X : d(x, x_0) < r\}.$$

## Example

In $\mathbb{R}$ with the usual norm (absolute value), open balls are symmetric open intervals, i.e.

$(x_0 - r,\ x_0 + r).$

# Example: Open ball in $\mathbb{R}^2$ with different metrics




(a) 1-norm (taxicab metric)

(b) 2-norm (Euclidean metric)

(c) $\infty$-norm

$d(x, y) = \sum |x - y|$

$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

$d(x, y) = \max_i |x_i - y_i|$

Figure: $B_r(0)$ for different metrics

# References

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