

# Module 4: Metric Spaces II

## Operational math bootcamp



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# Outline

- Open and closed sets
- Sequences
  - Cauchy sequences
  - subsequences

## Definition (Open and closed sets)

Let  $(X, d)$  be a metric space.

- A set  $U \subseteq X$  is *open* if for every  $x \in U$  there exists  $\epsilon > 0$  such that  $B_\epsilon(x) \subseteq U$ .
- A set  $F \subseteq X$  is *closed* if  $F^c := X \setminus F$  is open.

**Note:**

## Proposition

Let  $(X, d)$  be a metric space.

- ① Let  $A_1, A_2 \subseteq X$ . If  $A_1$  and  $A_2$  are open, then  $A_1 \cap A_2$  is open.
- ② If  $A_i \subseteq X$ ,  $i \in I$  are open, then  $\bigcup_{i \in I} A_i$  is open.

*Proof.* (1) Let  $A_1, A_2 \subseteq X$ . If  $A_1$  and  $A_2$  are open, then  $A_1 \cap A_2$  is open.

(2) If  $A_i \subseteq X$ ,  $i \in I$  are open, then  $\cup_{i \in I} A_i$  is open.

Using DeMorgan, we immediately have the following corollary:

### Corollary

*Let  $(X, d)$  be a metric space.*

- ① *Let  $A_1, A_2 \subseteq X$ . If  $A_1$  and  $A_2$  are closed, then  $A_1 \cup A_2$  is closed.*
- ② *If  $A_i \subseteq X$ ,  $i \in I$  are closed, then  $\bigcap_{i \in I} A_i$  is closed.*

## Definition (Interior and closure)

Let  $A \subseteq X$  where  $(X, d)$  is a metric space.

- The *closure* of  $A$  is  $\bar{A} :=$
- The *interior* of  $A$  is  $\overset{\circ}{A} :=$
- The *boundary* of  $A$  is  $\partial A :=$

## Example

Let  $X = (a, b] \subseteq \mathbb{R}$  with the ordinary (Euclidean) metric. Then

## Proposition

*Let  $A \subseteq X$  where  $(X, d)$  is a metric space. Then  $\overset{\circ}{A} = A \setminus \partial A$ .*

*Proof.*

## Proposition

Let  $(X, d)$  be a metric space and  $A \subseteq X$ .  $\overline{A}$  is closed and  $\overset{\circ}{A}$  is open.

*Proof.*

## Remark

In fact,  $\overset{\circ}{A} = \bigcup \{U : U \text{ is open and } U \subseteq A\}$  and  $\overline{A} = \bigcap \{F : F \text{ is closed and } A \subseteq F\}$ .

# Sequences

## Definition (Sequence)

Let  $(X, d)$  be a metric space. A *sequence* is an ordered list of points  $x_n$ ,  $n \in \mathbb{N}$ , in  $X$ , denoted  $(x_n)_{n \in \mathbb{N}}$ . We say that a sequence  $(x_n)_{n \in \mathbb{N}}$  *converges* to a point  $x \in X$  if

**Recall:**  $\overline{A} =$

### Proposition

Let  $(X, d)$  be a metric space, and let  $A \subseteq X$ . Then  $\overline{A}$  is equal to the set of points in  $X$  which are limits of a sequence in  $A$ .

*Proof.*

### Corollary

A set  $F \subseteq X$ , where  $(X, d)$  is a metric space, is closed if and only if every sequence in  $F$  which converges in  $X$  converges to a point in  $F$ .

**Remark:**

# Cluster points of a set

## Definition

Let  $(X, d)$  be a metric space and  $A \subseteq X$ . A point  $x \in X$  is a *cluster point* of  $A$  (also called accumulation point) if for every  $\epsilon > 0$ ,  $B_\epsilon(x)$  contains infinitely many points in  $A$ .

## Proposition

$x \in X$  is a cluster point of  $A \subseteq X$  where  $(X, d)$  is a metric space if and only if there exists a sequence of points  $x_n \in A$ ,  $n \in \mathbb{N}$ , such that  $x_n \rightarrow x$ .

*Proof.*

Combining the previous result with the limit characterization of closure gives the following:

### Corollary

For  $A \subseteq X$ ,  $(X, d)$  a metric space, we have

$$\overline{A} = A \cup \{x \in X : x \text{ is a cluster point of } A\}.$$

# Cauchy sequences

## Definition (Cauchy sequence)

Let  $(X, d)$  be a metric space. A sequence denoted  $(x_n)_{n \in \mathbb{N}} \in X$  is called a *Cauchy sequence* if

## Proposition

Let  $(X, d)$  be a metric space, and let  $(x_n)_{n \in \mathbb{N}}$  be a convergent sequence in  $X$ . Then  $(x_n)_{n \in \mathbb{N}}$  is Cauchy.

*Proof.*

## Definition

A metric space where every Cauchy sequence converges (to a point in the space) is called *complete*.

## Example:

## Proposition

Let  $(X, d)$  be a metric space, and let  $Y \subseteq X$ .

- (i) If  $X$  is complete and if  $Y$  is closed in  $X$ , then  $Y$  is complete.
- (ii) If  $Y$  is complete, then it is closed in  $X$ .

*Proof.*

# Subsequences

## Definition

Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in a metric space  $(X, d)$ . Let  $(n_k)_{k \in \mathbb{N}}$  be a sequence of natural numbers with  $n_1 < n_2 < \dots$ . The sequence  $(x_{n_k})_{k \in \mathbb{N}}$  is called a *subsequence* of  $(x_n)_{n \in \mathbb{N}}$ . If  $(x_{n_k})_{k \in \mathbb{N}}$  converges to  $x \in X$ , we call  $x$  a *subsequential limit*.

## Example

$$((-1)^n)_{n \in \mathbb{N}}$$

## Proposition

A sequence  $(x_n)_{n \in \mathbb{N}}$  in a metric space  $(X, d)$  converges to  $x \in X$  if and only if every subsequence of  $(x_n)_{n \in \mathbb{N}}$  also converges to  $x$ .

*Proof.*

*Proof continued*

# References

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