Module 4: Metric Spaces II

Operational math bootcamp

Statistical Sciences
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△ metric d: function satisfies

1) $d(x,y) \geq 0$ and $d(x,y)=0$ iff $x=y$

2) $d(x,y) = d(y,x)$

3) Triangle inequality : $d(x,z) \leq d(x,y) + d(y,z)$

○ p- norms

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

$p=1$, $\|x\|_1 = \sum_{i=1}^{n} |x_i|$

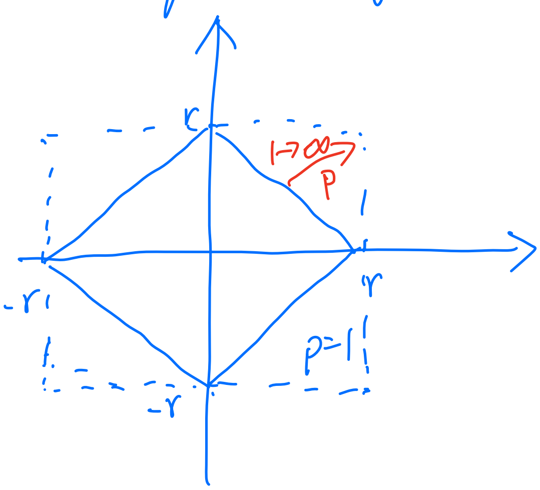
$p=2$, $\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$ Euclidean distance

$p \to \infty$, $\|x\|_\infty = \max |x_i|$ sup-norm

open ball, (defined with norms).

Consider an open ball in $\mathbb{R}^2$.

$$B_r(x,y) = \{ (x,y) \in \mathbb{R}^2, \|(x,y)\| < r \}$$



when $p=1$,

$$B_r(x,y) = \{ (x,y) \in \mathbb{R}^2, |x| + |y| < r \}$$

when $p=\infty$.

when $p: 2 \to \infty$.

Outline

- Open and closed sets
- Sequences
    - Cauchy sequences
    - subsequences

$B_\varepsilon(x) = \{ y \in X \mid d(x,y) < \varepsilon \}$, a ball in (X,d)

Definition (Open and closed sets)

Let (X, d) be a metric space.

- A set $U \subseteq X$ is *open* if for every $x \in U$ there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$.
- A set $F \subseteq X$ is *closed* if $F^c := X \setminus F$ is open.

x, $B_\varepsilon(x) \subseteq U$, U

e.g. $\mathbb{R}$, $[0,1]$, for point 1, $\forall B_\varepsilon(1)$, $B_\varepsilon(1) \not\subseteq [0,1]$

Note: For empty set ϕ, X, we define it both open and closed. $\phi = X \setminus X$

Proposition

Let (X, d) be a metric space.

1. *Let $A_1, A_2 \subseteq X$. If A_1 and A_2 are open, then $A_1 \cap A_2$ is open.* Intersection. finite
2. *If $A_i \subseteq X$, $i \in I$ are open, then $\cup_{i \in I} A_i$ is open.* Union, infinite

I can be infinite

e.g. We don't have infinite intersection,
counter example: $\{(-\frac{1}{n}, \frac{1}{n})\}_{n=1}^{\infty}$, $\bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}) = \{0\}$, For any singleton set,

it's closed.

Proof. (1) Let $A_1, A_2 \subseteq X$. If A_1 and A_2 are open, then $A_1 \cap A_2$ is open.

Let $x \in A_1 \cap A_2$

Since both A_1 and A_2 are open, $\exists\, \varepsilon_1, \varepsilon_2 > 0$, s.t. $B_{\varepsilon_1}(x) \subset A_1$, $B_{\varepsilon_2}(x) \subset A_2$

Let $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$. Then $B_\varepsilon(x) \subset B_{\varepsilon_1}(x) \cap B_{\varepsilon_2}(x) \subset A_1 \cap A_2$

So by definition of open set, $A_1 \cap A_2$ is open.

(2) If $A_i \subseteq X$, $i \in I$ are open, then $\cup_{i \in I} A_i$ is open.

Let $x \in \bigcup_{i \in I} A_i$, $\exists\, i \in I$ s.t. $x \in A_i$

since A_i open, $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset A_i \subset \bigcup_{i \in I} A_i$

So by definition of open set, $\bigcup_{i \in I} A_i$ is open.

Using DeMorgan, we immediately have the following corollary:

Corollary

Let (X, d) be a metric space.

1. *Let $A_1, A_2 \subseteq X$. If A_1 and A_2 are closed, then $A_1 \cup A_2$ is closed.* Union, Finite.
2. *If $A_i \subseteq X$, $i \in I$ are closed, then $\cap_{i \in I} A_i$ is closed.* Intersection Infinite

	Union	Intersection
Open set	Infinite	Finite
Closed set	Finite	Infinite

Definition (Interior and closure)

Let $A \subseteq X$ where (X, d) is a metric space.

- The *closure* of A is $\overline{A} := \{x \in X \mid \forall \varepsilon > 0, B_\varepsilon(x) \cap A \neq \phi\}$ / $A \subseteq \overline{A}$
- The *interior* of A is $\mathring{A} := \{x \in X \mid \exists \varepsilon > 0,$ s.t. $B_\varepsilon(x) \subseteq A\}$ / $\mathring{A} \subseteq A$
- The *boundary* of A is $\partial A := \{x \in X \mid \forall \varepsilon > 0$ s.t. $B_\varepsilon(x) \cap A \neq \phi$ and $B_\varepsilon(x) \cap A^c \neq \phi\}$. / $\partial A \subseteq \overline{A}$

Example

Let $X = (a, b] \subseteq \mathbb{R}$ with the ordinary (Euclidean) metric. Then

$\mathring{X} = (a, b)$, $\overline{X} = [a, b]$ $\partial X = \{a, b\}$

Intuition of closure / interior / boundary.

Closure: a point in $\bar{A}$ if it is either in A, or it can be approximated arbitrarily closely by points of A "from $B_\varepsilon(x)$ in definition"

Interior: a point in $\mathring{A}$ if it's safely inside A, not on the boundary

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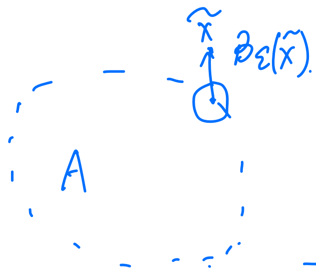
Closure

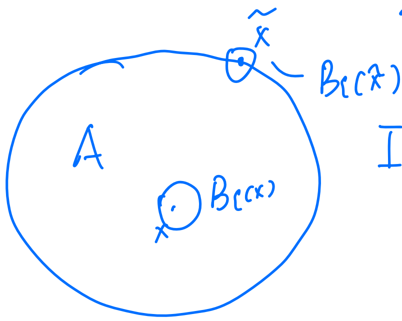


If $\tilde{x} \in \bar{A}$, $B_\varepsilon(\tilde{x})$ must intersect with A

$\tilde{x} \in \bar{A}$ / $x \in \bar{A}$ as well

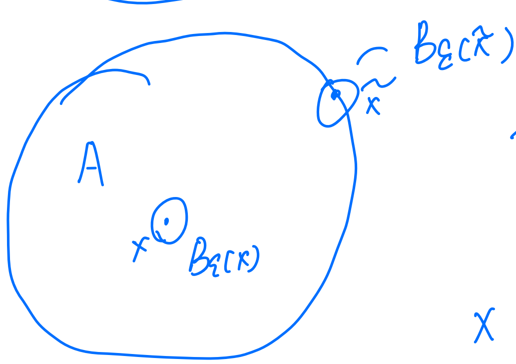
$\tilde{x} \notin \mathring{A}$, $B_\varepsilon(\tilde{x}) \nsubseteq A$

Interior



If $x \in \mathring{A}$, $B_\varepsilon(x) \subseteq A$

Boundary



$\tilde{x} \in \partial A$, $B_\varepsilon(\tilde{x})$ intersects both A and A^c

$x \notin \partial A$, $B_\varepsilon(x)$ doesn't intersect with A^c

Summary.

		Closure	Interior	Boundary.
x	(inside)	$\in$	$\in$	$\notin$
$\tilde{x}$	(boundary).	$\in$	$\notin$	$\in$

need to show "$\subset$" and "$\supset$" separately,

Proposition

Let $A \subseteq X$ where (X, d) is a metric space. Then $\mathring{A} = A \setminus \partial A$.

Proof. "$\subset$" part by contradiction

Let $x \in \mathring{A}$, Then $\exists\, \varepsilon > 0$, s.t. $B_\varepsilon(x) \subset A$

suppose $x \in \partial A$, Then by definition, $B_\varepsilon(x) \cap A^c \neq \phi$ $\Rightarrow$ contradict.

"$\supset$" part

Let $x \in A \setminus \partial A$. By definition of ∂A,

$\exists\, \varepsilon > 0$, s.t. $B_\varepsilon(x) \cap A = \phi$ or $B_\varepsilon(x) \cap A^c = \phi$

Since $x \in A$, we cannot have $B_\varepsilon(x) \cap A = \phi$

So $B_\varepsilon(x) \cap A^c = \phi$, thus $B_\varepsilon(x) \subset A$

So $x \in \mathring{A}$ Q.E.D.

Proposition

Let (X, d) be a metric space and $A \subseteq X$. $\overline{A}$ is closed and $\mathring{A}$ is open.

Proof. We'll prove the stronger version.

stronger version

Remark

In fact, $\mathring{A} = \bigcup\{U : U \text{ is open and } U \subseteq A\}$ and $\overline{A} = \bigcap\{F\colon F \text{ is closed and } A \subseteq F\}$.

the largest open set inside A

the smallest closed set containing A

$\boxed{\text{proof}}$ 1) $\mathring{A} = \cup \{ U \mid U \text{ is open and } U \subset A \}$

"$\subset$" part

let $x \in \mathring{A}$. Then $\exists\, \varepsilon > 0$, s.t. $B_\varepsilon(x) \subset A$

Since $B_\varepsilon(x)$ is open, we can let $U = B_\varepsilon(x)$

so we have $x \in \cup \{ U \mid U \text{ open and } U \subset A \}$

"$\supset$" part let $x \in \cup \{ U \mid U \text{ open and } U \subset A \}$

Then, $\exists\, U$ is open, s.t. $x \in U$, and $U \subset A$

Since U is open, by definition of open set

$\exists\, \varepsilon > 0$, s.t. $B_\varepsilon(x) \subset U \subset A \Rightarrow B_\varepsilon(x) \subset A$

$\Rightarrow x \in \mathring{A}$

2). $\bar{A} = \cap \{ F \mid F \text{ is closed and } A \subset F \}$

"$\subset$" part $A \subseteq \bar{A}$, so it suffices to prove $\bar{A}$ is closed.

$x \in \bar{A}^c$. Then by definition of $\bar{A}$, $\exists\, \varepsilon > 0$, s.t. $B_\varepsilon(x) \cap A = \phi$.

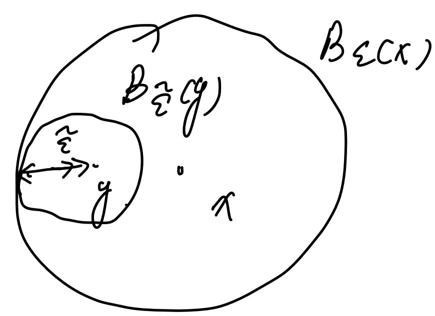
That means $B_\varepsilon(x) \subset A^c$

We need to further show $B_\varepsilon(x) \subset \bar{A}^c$

Let $y \in B_\varepsilon(x)$ and define $\tilde{\varepsilon} = \varepsilon - d(x,y)$.

Then $B_{\tilde{\varepsilon}}(y) \subset B_\varepsilon(x) \subset A^c$

By triangle inequality



$\therefore y \in \bar{A}^c$

Thus $B_\varepsilon(x) \subset \bar{A}^c$, which implies $\bar{A}$ is a closed set.

"$\supset$" part   we've known $\bar{A}$ is closed.

Let F be closed $F \supset A \Leftrightarrow F^c \subset A^c$

we must show $\bar{A} \subset F \Leftrightarrow F^c \subset \bar{A}^c$

Let $x \in F^c \subset A^c$. Since F^c is open,

$\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset F^c \subset A^c$

So $B_\varepsilon(x) \cap A = \emptyset$

By definition of $\bar{A}$, $x \notin \bar{A} \Leftrightarrow x \in \bar{A}^c$

so $F^c \subseteq \bar{A}^c \Leftrightarrow \bar{A} \subset F$

Sequences

Definition (Sequence)

Let (X, d) be a metric space. A *sequence* is an ordered list of points x_n, $n \in \mathbb{N}$, in X, denoted $(x_n)_{n\in\mathbb{N}}$. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ *converges* to a point $x \in X$ if

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $d(x_n, x) < \varepsilon$, $\forall n \geq N$

Recall: $\overline{A} = \{x \in X \mid \forall \varepsilon > 0,\ B_\varepsilon(x) \cap A \neq \emptyset\}$

Proposition

Let (X, d) be a metric space, and let $A \subseteq X$. Then $\overline{A}$ is equal to the set of points in X which are limits of a sequence in A.

Proof. $\overline{A} = \{x \in X \mid \exists \{x_n\} \subset A, \text{ s.t. } x_n \to x\}$

" "$\subset$" " part

(open ball language)

Let $x \in \overline{A}$. By definition, we have $\forall \varepsilon > 0$, $B_\varepsilon(x) \cap A \neq \emptyset$

Let $\varepsilon = 1/n$, Then $B_{1/n}(x) \cap A \neq \emptyset$.

Just pick any (one) $x_n \in B_{1/n}(x) \cap A$,

Then $x_n \in A$ and $d(x_n, x) < 1/n$

For $\forall \varepsilon > 0$, by taking $1/n_\varepsilon < \varepsilon \iff n_\varepsilon > \varepsilon^{-1}$

then for any $n \geq n_\varepsilon$

$$d(x_n, x) < 1/n \leq 1/n_\varepsilon < \varepsilon$$

"$\supset$" part

Let x be the limit of $\{x_n\} \subset A$

$\forall \varepsilon > 0$, $\exists n_\varepsilon$ s.t. $n \geq n_\varepsilon \Rightarrow d(x_n, x) < \varepsilon \Leftrightarrow x_n \in B_\varepsilon(x)$

Since $\{x_n\} \subset A$, $x_n \in B_\varepsilon(x) \cap A$

Thus $B_\varepsilon(x) \cap A \neq \emptyset$. Therefore $x \in \overline{A}$

$\bar{A}$ = the smallest closed set containing A

If A is closed, then naturally A is the smallest closed set containing A

so $A = \bar{A}$

If $A = \bar{A}$, then we have A is closed.

Corollary

A set $F \subseteq X$, where (X, d) is a metric space, is closed if and only if every sequence in F which converges in X converges to a point in F.

will converge to the point in $\bar{F}$

Remark: F is closed $\iff F = \bar{F}$

$\iff F = \{x \in X \mid \exists \{x_n\} \subseteq F \text{ s.t. } x_n \to x\}$

Cluster points of a set

Definition

Let (X, d) be a metric space and $A \subseteq X$. A point $x \in X$ is a *cluster point* of A (also called accumulation point) if for every $\epsilon > 0$, $B_\epsilon(x)$ contains infinitely many points in A.

x is cluster point of A

$\iff \forall \varepsilon > 0, \; B_\varepsilon(x) \cap A \neq \emptyset$

Proposition

$x \in X$ is a cluster point of $A \subseteq X$ where (X, d) is a metric space if and only if there exists a sequence of points $x_n \in A$, $n \in \mathbb{N}$, such that $x_n \to x$.

Proof. cluster point = convergence point.

" $\Rightarrow$ " let x be cluster point.

$\forall n$, let $\varepsilon_n = 1/n$. Then $B_{\varepsilon_n}(x) \cap A \neq \emptyset \Rightarrow$

we can pick $x_n \in B_{\varepsilon_n}(x) \cap (A \setminus \{x\})$

Then $d(x_n, x) < 1/n$ and $x_n \in A$

$\hookrightarrow x_n \to x$ as $n \to \infty$

" $\Leftarrow$ " part.

$\forall \varepsilon > 0$, $\exists n_\varepsilon$ s.t. $n \geq n_\varepsilon \Rightarrow d(x_n, x) < \varepsilon$

This means $x_n \in B_\varepsilon(x)$.

Thus, $x_n \in B_\varepsilon(x) \cap A$

$\Rightarrow B_\varepsilon(x) \cap A \neq \emptyset$

Therefore x is a cluster point.

Recall $\bar{A} = \{x \in X \mid \exists \{x_n\} \subset A, \text{ s.t. } x_n \to x\}$

Combining the previous result with the limit characterization of closure gives the following:

Corollary

For $A \subseteq X$, (X, d) a metric space, we have

$$\overline{A} = A \cup \{x \in X : x \text{ is a cluster point of } A\}.$$

Cauchy sequences

Convergence of a sequence:

$\forall \varepsilon > 0, \exists M$ s.t. $n \geq M \Rightarrow d(x_n, x) < \varepsilon$

series limit

Definition (Cauchy sequence)

Let (X, d) be a metric space. A sequence denoted $(x_n)_{n \in \mathbb{N}} \in X$ is called a Cauchy sequence if

$\forall \varepsilon > 0, \exists M$ s.t. $n_1, n_2 \geq M \Rightarrow d(x_{n_1}, x_{n_2}) < \varepsilon$.

series point

△ Convergence of x_n is not guaranteed.

△ Convergence $\xrightarrow{\quad} $ Cauchy sequence.

Convergence $\cancel{\xleftarrow{\quad}}$ Cauchy sequence.

Proposition

Let (X, d) be a metric space, and let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in X. Then $(x_n)_{n\in\mathbb{N}}$ is Cauchy.

Proof. Let x be the limit of $\{x_n\}$.

$\forall \varepsilon > 0, \exists M$, s.t. $n \geq M \Rightarrow d(x_n, x) < \varepsilon$

Then $\forall n_1, n_2 \geq M$, by triangle inequality

$$d(x_1, x_2) \leq d(x_1, x) + d(x, x_2) < \varepsilon + \varepsilon = 2\varepsilon$$

$\triangle$ $\mathbb{Q}$ rational space. Let $x_1 = 3.1$, $x_2 = 3.14$, $x_3 = 3.141$, $x_4 = 3.1415$

looks like "$x_n \to \pi$"

we can show $\{x_n\}$ Cauchy sequence.

However, $\{x_n\}$ is not a convergent sequence.

Because π is not inside $\mathbb{Q}$.

Definition

A metric space where every Cauchy sequence converges (to a point in the space) is called *complete*.

Example: $\mathbb{R}$, $\mathbb{R}^n$ with usual euclidean metric are complete

$\mathbb{Q}$ is not complete

Proposition

Let (X, d) be a metric space, and let $Y \subseteq X$.

(i) If X is complete and if Y is closed in X, then Y is complete.

(ii) If Y is complete, then it is closed in X.

Proof. 1) $Y \subseteq X$, Y is closed, X is complete $\Rightarrow$ Y is complete

Let $\{x_n\} \subset Y$ be cauchy. since $\{x_n\} \subset Y \subset X$, and X is complete

$\exists x \in X$, $x_n \to x$. Since Y is closed, Y must contain any convergence limit.

That is saying, $x \in Y$.

2) $Y \subseteq X$, Y is complete, X is complete, $\Rightarrow$ Y is closed.

Since Y is complete, any convergent sequence in Y

converges to a point in Y.

That's equivalent to say that Y is closed.

Subsequences

Definition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space (X, d). Let $(n_k)_{k\in\mathbb{N}}$ be a sequence of natural numbers with $n_1 < n_2 < \cdots$. The sequence $(x_{n_k})_{k\in\mathbb{N}}$ is called a *subsequence* of $(x_n)_{n\in\mathbb{N}}$. If $(x_{n_k})_{k\in\mathbb{N}}$ converges to $x \in X$, we call x a *subsequential limit*.

Example

$((-1)^n)_{n\in\mathbb{N}}$ do not have a limit

$\hookrightarrow$ $n = 2m$, $\cdot$ $(-1)^{2m} = 1$, limit 1.

$\hookrightarrow$ $n = 2m+1$, $(-1)^{2m+1} = -1$, limit -1

Proposition

A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X, d) converges to $x \in X$ if and only if every subsequence of $(x_n)_{n\in\mathbb{N}}$ also converges to x.

Proof. $\Longrightarrow$ trivial.

$\Longleftarrow$ Suppose x_n doesn't converge x. $\forall \varepsilon, \exists M$ $n \geq M$, $d \not< \varepsilon$.

$\exists \varepsilon, \forall N, \exists n \geq N$, $d(x_n, x) \geq \varepsilon$

Using this, we can construct a subsequence x_{n_k}

s.t. $d(x_{n_k}, x) \geq \varepsilon$ for all k.

This subsequence cannot converge to x. $\Longrightarrow$ contradiction.

Proof continued

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