Module 5: Metric spaces III

Operational math bootcamp

Statistical Sciences
UNIVERSITY OF TORONTO

University of Toronto

July 13, 2025

Last time

Looked at open and closed sets.

Discussed sequences, which includes Cauchy sequences and subsequences:

- Convergent sequence: $x_n \to x \Leftrightarrow \forall \epsilon > 0 \ \exists\, n_\epsilon \in \mathbb{N}$ s.t. $d(x_n, x) < \epsilon$ for all $n \geq n_\epsilon$
- Cauchy sequence: $\forall \epsilon > 0 \ \exists\, n_\epsilon \in \mathbb{N}$ s.t. $d(x_n, x_m) < \epsilon$ for all $n, m \geq n_\epsilon$
- Proved that a convergent sequence is Cauchy
- Discussed complete metric spaces, where all Cauchy sequences converge (like $\mathbb{R}$ with the usual metric, absolute value)
- Proved that a sequence converges to x if and only if all subsequences converge to x

cauchy sequence $\Longleftarrow$ / $\not\Longrightarrow$ convergent sequence

Define metric space is complete. if all cauchy sequence is convergent

Open set. (X, d)

$U \subseteq X$ is open if $\forall x \in U$, $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq U$

Closed set

$F \subseteq X$ is closed if $F^c := X \setminus F$ is open

Outline for today

Metric d $\to$ distance $\to$ convergence

$\downarrow$

Continuity.

- Continuity
- Equivalent metrics
- Density and separability

Continuity

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces, let $x_0 \in X$, and let $f: X \to Y$. f is *continuous* at x_0 if for every sequence $(x_n)_{n\in\mathbb{N}}$ in X that converges to x_0, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.

We say that f is continuous (in X) if it is continuous at every point in X.

f is cts in x_0

$\forall$ $x_n \to x_0$, we have $f(x_n) \to f(x_0)$

Theorem

Let (X, d_X) and (Y, d_Y) be metric spaces, let $x_0 \in X$, and let $f : X \to Y$. The following are equivalent: convergent sequence.

(i) *f is continuous at x_0* – definition in last page.

(ii) *for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0))) < \epsilon$ for all $x \in X$ with $d_X(x, x_0) < \delta$* – metric

(iii) *for each $\epsilon > 0$, there is $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$* open ball.

(i) → (ii) → (iii) → (i)

(iv) $\forall\, U \subseteq Y$, and U is open, then $f^{-1}(U)$ is still open

(v) $\forall\, F \subseteq Y$, and F is closed, then $f^{-1}(F)$ is still closed.

(i) f is continuous at x_0

(ii) for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0))) < \epsilon$ for all $x \in X$ with $d_X(x, x_0) < \delta$

(iii) for each $\epsilon > 0$, there is $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$

Proof. (i) $\Rightarrow$ (ii) Contradiction

Suppose (ii) is false. $\exists \varepsilon > 0$, $\forall \delta > 0$, s.t. $d_X(x, x_0) < \delta$ and $d_Y(f(x), f(x_0)) \geq \varepsilon$

contradiction

Then $\forall n \in \mathbb{N}$, taking $\delta = 1/n$, $\exists x_n$ s.t. $d_X(x_n, x_0) < 1/n$ and $d_Y(f(x_n), f(x_0)) \geq \varepsilon$

Since $x_n \to x_0$ (because $d_X(x_n, x_0) < 1/n$, $n \to \infty$), by (i), we have $f(x_n) \to f(x_0)$

(ii) $\Rightarrow$ (iii) (ii) says $\forall \varepsilon>0, \exists \delta>0$ s.t. $\underbrace{x \in \{x \in X \mid d_X(x, x_0) < \delta\}}_{x \in B_\delta(x_0)}$ implies $\underbrace{d_Y(f(x), f(x_0)) < \varepsilon}_{f(x) \in B_\varepsilon(f(x_0))}$

$\Leftrightarrow \forall \varepsilon>0, \exists \delta>0$, s.t. $x \in B_\delta(x_0) \Rightarrow \underbrace{f(x) \in B_\varepsilon(f(x_0))}$ $\quad x \in f^{-1}(B_\varepsilon(f(x_0)))$

$\Leftrightarrow \forall \varepsilon>0, \exists \delta>0$, s.t. $B_\delta(x_0) \subseteq f^{-1}(B_\varepsilon(f(x_0)))$ $\quad \underbrace{x \in B_\delta(x_0) \Rightarrow x \in f^{-1}(B_\varepsilon(f(x_0)))}_{\subseteq}$

(iii) $\Rightarrow$ (i) Let $x_n \to x_0$

By (iii), $\forall \varepsilon>0, \exists \delta>0$ s.t. $B_\delta(x_0) \subseteq f^{-1}(B_\varepsilon(f(x_0)))$

Since $x_n \to x_0$, (by definition of convergence), $\exists n_\delta$, s.t. $x_n \in B_\delta(x_0)$ if $n \geq n_\delta$.

$\subseteq f^{-1}(B_\varepsilon(f(x_0)))$

Therefore, $x_n \in f^{-1}(B_\varepsilon(f(x_0))) \Leftrightarrow f(x_n) \in B_\varepsilon(f(x_0))$

$\Leftrightarrow d_Y(f(x_0), f(x_n)) < \varepsilon$ $\quad$ definition of open ball.

Thus, we are able to show

$\forall \varepsilon>0, \exists n_\delta$, s.t. $d_Y(f(x_n), f(x_0)) < \varepsilon$ if $n \geq n_\delta$. that is

$f(x_n) \to f(x_0)$

Corollary

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f: X \to Y$. The following are equivalent:

(i) f is continuous

(ii) if $U \subseteq Y$ is open, then $f^{-1}(U)$ is open $\leftarrow$ standard way to define continuity for a more general way.

(iii) if $F \subseteq Y$ is closed, then $f^{-1}(F)$ is closed

pre-image of open set is still open

(i) $\rightleftarrows$ (ii) $\rightleftarrows$ (iii)

We need the following results about sets and functions:

Let X and Y be sets and $f: X \to Y$. Let $A, B \subseteq Y$. Then

$f(A^c) \neq [f(A)]^c$

1. $A \subseteq B \implies f^{-1}(A) \subseteq f^{-1}(B)$ $\leftarrow$ by definition
2. $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$ $\leftarrow$ by definition

$\triangle$ we don't have $\underline{f(Y \setminus A) = X \setminus f(A)}$

$f^{-1}(A^c) = [f^{-1}(A)]^c$ complement of pre-image = pre-image of complement

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces and let $f: X \to Y$.

(i) $\Rightarrow$ (ii): Let $U \subseteq Y$ be open, we want to show $f^{-1}(U)$ is open.

$\boxed{\text{(i) } x_n \to x_0,\ f(x_n) \to f(x_0)}$

Let $x \in f^{-1}(U)$ since $f(x) \in U$, U is open,

$\exists \varepsilon > 0$ s.t. $B_\varepsilon(f(x)) \subset U \Rightarrow \underline{f^{-1}(B_\varepsilon(f(x))) \subset f^{-1}(U)}$

By (iii) in the previous theorem, $\exists \delta > 0$ s.t. $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x))) \subset \underline{f^{-1}(U)}$

$\forall x \in f^{-1}(U)$, $B_\delta(x) \subset f^{-1}(U)$, so $f^{-1}(U)$ must open.

Summary.

what we have is (i) $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$ (*)

(iii) in the previous theorem,

$\exists\ \delta > 0$ s.t. $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))$ (**)

what we need to prove (ii) $U \subseteq Y$ is open, $f^{-1}(U)$ is open.

$\Leftarrow$. Consider arbitrary open U, and prove that $\underline{f^{-1}(U) \text{ is open}}$.

$\Leftarrow$. Consider arbitrary $x \in f^{-1}(U)$, and show that $\exists\ \varepsilon/\delta$, $B_\delta(x) \subseteq f^{-1}(U)$.

we need the help of (*) / (**).

What we have is that $\forall U \subseteq Y$, $f^{-1}(U)$ is still open.

What we need to prove is that, $\{$ $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$

$\forall \varepsilon > 0$, $\exists \delta > 0$, s.t.

$B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))$

(ii) $\Rightarrow$ (i)

$\forall x \in X$, $\forall \varepsilon > 0$, $B_\varepsilon(f(x))$ is open set in Y

By (ii), we have $f^{-1}(B_\varepsilon(f(x)))$ is open.

By definition of open set, $\exists \delta > 0$, s.t. $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))$

Since this holds for $\forall x \in X$, f is cts.

(ii) $\Rightarrow$ (iii)

(ii) $U \subseteq Y$ is open then $f^{-1}(U)$ is still open

(iii) $F \subseteq Y$ is closed set, $f^{-1}(F)$ is still closed

Let $F \subset Y$ be closed, (by the def of closed set)

F^c is open $\xrightarrow{(ii)}$ $f^{-1}(F^c)$ is open

Thus we have $[f^{-1}(F^c)]^c = f^{-1}(F)$ is closed

property: $[f^{-1}(F)]^c = f^{-1}(F^c)$

(iii) $\Rightarrow$ (ii)

Let $U \subset Y$ be open, since U^c is closed

$f^{-1}(U^c)$ is closed (by (iii)).

$f^{-1}(U^c) = [f^{-1}(U)]^c$, which is closed

$\Rightarrow$ $f^{-1}(U)$ is open

Recall: f is cts in x_0, $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon$

f is cts in X, for one x_0, we would have one $\delta_{\varepsilon, x_0}$

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$.

- f is *uniformly continuous* if for all $\epsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for every $x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$, we have $d_Y(f(x_1), f(x_2))) < \epsilon$ defined in whole set.
- f is *Lipschitz continuous* if there exists a $K > 0$ such that for every $x_1, x_2 \in X$ we have $d_Y(f(x_1), f(x_2))) \leq K d_X(x_1, x_2)$

known constant.

Proposition

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$.

$$f \text{ is Lipschitz continuous} \Rightarrow \text{f is uniformly continuous} \Rightarrow \text{f is continuous}$$

Proof is one of your exercises.

Contraction Mapping Theorem

Definition

Let (X, d) be a metric space and let $f: X \to X$. We say that $x^* \in X$ is a *fixed point* of f if $f(x^*) = x^*$.

Definition

Let (X, d) be a metric space and let $f: X \to X$. f is a *contraction* if there exists a constant $k \in [0, 1)$ such that for all $x, y \in X$, $d(f(x), f(y)) \leq kd(x, y)$.

Observe that a function is a contraction if and only if it is Lipschitz continuous with constant $K < 1$.

if not complete, cauchy sequence can be not convergent

Theorem (Contraction Mapping Theorem)

Suppose that $f: X \to X$ is a contraction and the metric space X is complete. Then f has a unique fixed point x^*.

$\exists !$

Example

Let $f\colon \left[-\frac{1}{3}, \frac{1}{3}\right] \to \left[-\frac{1}{3}, \frac{1}{3}\right]$ be defined by the mapping $x \mapsto x^2$. Assume we use the standard Euclidean metric, $d(x, y) = |x - y|$. f has a unique fixed point because

① X is complete { closed subset of a complete space $(\mathbb{R}, d)$.

②. f is Lip-cts with $k < 1$

$\forall\, x, y \in [-1/3, 1/3]$, $\quad f(x) - f(y) = x^2 - y^2 = (x-y)(x+y)$

$|f(x) - f(y)| = |x-y|\,|x+y| \le 2/3\,|x-y| \quad , \; k = 2/3 < 1$

Equivalent metrics

$x_n,$

Definition (Equivalent metrics)

Two metrics d_1 and d_2 on a set X are *equivalent* if the identity maps from (X, d_1) to (X, d_2) and from (X, d_2) to (X, d_1) are continuous.

$d_1(x_n, x) \to 0 \iff d_2(x_n, x) \to 0$ / $x_n \to x$ under d_1 $\iff$ $x_n \to x$ under d_2

Proposition

Two metrics d_1, d_2 on a set X are equivalent if and only if they have the same open sets or the same closed sets.

$U \subseteq X$, U is open in (X, d_1) $\rightleftarrows$ U is open in (X, d_2)

Recall the order of introducing concepts.

Set X $\longrightarrow$ Metric d $\longrightarrow$ Metric space (X, d)

$\downarrow$

open / closed set U

[is defined in metric space $(X, \underline{d})$]

Definition

Two metrics d_1 and d_2 on a set X are *strongly equivalent* if for every $x, y \in X$, there exists constants $\alpha > 0$ and $\beta > 0$ such

$$\alpha d_1(x, y) \leq d_2(x, y) \leq \beta d_1(x, y).$$

$$\begin{cases} d_2(x,y) \leq \beta d_1(x,y) \\ d_1(x,y) \leq \frac{1}{\alpha} d_2(x,y) \end{cases}$$

If two metrics are strongly equivalent then they are equivalent. The proof of this is one of the exercises.

$\{$ d_1 can be bounded by d_2
d_2 can be bounded by d_1

Example

We show that the Euclidean distance (induced by 2-norm) and the metric induced by the ∞-norm are equivalent on $\mathbb{R}^n$.

we can show these two are strongly equivalent.

$\|x-y\|_\infty = \max_i |x_i - y_i|$, $\|x-y\|_2 = \sqrt{\sum_{i=1}^n (x_i-y_i)^2}$.

$\|x-y\|_\infty \le \sqrt{\sum_{i=1}^n (x_i-y_i)^2} = 1\, \|x-y\|_2$

$\|x-y\|_2 \le \sqrt{\sum_{i=1}^n \max_i (x_i-y_i)^2} = \sqrt{n} \cdot \|x-y\|_\infty$ $\Rightarrow \|x-y\|_\infty \le \|x-y\|_2 \le \sqrt{n}\, \|x-y\|_\infty$

Can you think of an example that we've seen of a metric that isn't equivalent to the Euclidean metric?

Let X be a set and define $d \colon X \times X \to \mathbb{R}$ by (without loss of generality, assume $X = \mathbb{R}$)

$$d(x,y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

All three properties are satisfied for metric

With such metric, every singleton is open set.

e.g. $\forall x \in X$, $B_{1/2}(x) = \{x\}$

(X, d)

That is saying, d on X has different open sets with Euclidean metric on X.

$\Rightarrow$ They are not equivalent

Density

Definition

Let (X, d) be a metric space. A subset $A \subseteq X$ is called *dense* if $\overline{A} = X$.

Recall $\overline{A} = \{x \in X \mid \forall \varepsilon,\ B_\varepsilon(x) \cap A \neq \emptyset\}$.

$A \subset X$ is dense $\Longleftrightarrow \forall x \in X,\ \forall \varepsilon > 0,\ B_\varepsilon(x) \cap A \neq \emptyset$

Why are dense sets important?

- We can approximate any element of a space using elements from a smaller subset.

Examples

1. $(\mathbb{R}, |\cdot|)$ standard Euclidean

   $\mathbb{Q}$ = {all rational number}.

   $\mathbb{Q}$ is dense. $\forall x \in \mathbb{R}/\mathbb{Q}$, x is irrational.

   we can approximate any x by rational number.

   For π, 3.14159...

2. Let X be a set and define $d \colon X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

The only dense set is X itself (exercise)

Definition

A metric space (X, d) is *separable* if it contains a countable dense subset.

Example:

$\mathbb{Q} \subset \mathbb{R}$, Then $(\mathbb{R}, d)$ is separable

dense and countable.

Is there any metric space is not separable?

Example

Define $\ell_\infty = \{(x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{R}, \sup_{n\in\mathbb{N}} |x_n| < \infty\}$, the space of bounded real valued sequences. Endow ℓ_∞ with a metric induced by the supremum norm, namely d.
$d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sup_{n\in\mathbb{N}} |x_n - y_n|$. Then ℓ_∞ is **not separable** with respect to the topology induced by this metric.

X

dense countable.

Note: this proof relies on content from the cardinality section. Specifically, Cantor's theorem gives us that $|A| < |\mathcal{P}(A)|$ for any set A.

Proof. ① Assume $S \subseteq \ell_\infty$ is countable, show it cannot be dense

$S = \{ (x_n^k)_{n\in\mathbb{N}} \mid k \in \mathbb{N} \}$.

Define a new sequence $(y_n)_{n\in\mathbb{N}}$ by

$$y_n = \begin{cases} 0 & \text{if } |x_n^n| > 1 \\ 2 & \text{if } |x_n^n| \le 1 \end{cases}$$

Proof continued.

By construction, $\forall n$, $|y_n| \le 2$. Thus $(y_n)_{n\in\mathbb{N}} \in l_\infty$

Furthermore, by construction of y_n, we have

$$d\left((y_n)_{n\in\mathbb{N}},\ (x_n^k)_{n\in\mathbb{N}}\right) \ge 1$$

element in S

$$\begin{array}{lccccc}
k=1 & x_1^1 & x_2^1 & x_3^1 & x_4^1 & \cdots \\
k=2 & x_1^2 & x_2^2 & x_3^2 & x_4^2 & \cdots \\
k=3 & x_1^3 & x_2^3 & x_3^3 & x_4^3 & \cdots \\
k=4 & x_1^4 & x_2^4 & x_3^4 & x_4^4 & \cdots
\end{array}$$

By taking $\varepsilon = 1/2$, then $B_{1/2}((y_n)_{n\in\mathbb{N}}) \cap S = \emptyset$

So S is not dense.

Contradicts.

Assume S is dense, show that it cannot be countable

For any subset $I \subset \mathbb{N}$, we can define $e^I = (e_n^I)_{n \in \mathbb{N}} \in \ell_\infty$ as

$$e_n^I = \begin{cases} 1 & , n \in I \\ 0 & , n \notin I \end{cases}$$

Note, if $I \neq J$, $\exists n$ s.t. $|e_n^I - e_n^J| = 1$

So, $d(e^I, e^J) = 1$ if $I \neq J$

Let $\varepsilon = 1/4$, $B_\varepsilon(e^I) \cap B_\varepsilon(e^J) = \emptyset$ if $I \neq J$

So all $B_\varepsilon(e^I)$ are disjoint for any $I \subset \mathbb{N}$

Let $S \subset \ell_\infty$ be a dense subset.

By definition of dense subset, $B_\varepsilon(e^I) \cap S \neq \emptyset$.

$\Rightarrow |S| \geq |\{\{I \mid I \subset \mathbb{N}\}\}|$

To show this, we need construct a injective function between S and $\{\{I \mid I \subset \mathbb{N}\}\}$

$= |P(\mathbb{N})| > |\mathbb{N}|$

$|S| > |\mathbb{N}|$

Therefore S cannot be countable.

References

Jiří Lebl (2022). *Basic Analysis I*. Vol. 1. Introduction to Real Analysis. https://www.jirka.org/ra/realanal.pdf

Runde, Volker (2005). *A Taste of Topology*. Universitext. url: https://link.springer.com/book/10.1007/0-387-28387-0