Module 6: Metric Spaces IV

Operational math bootcamp

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Outline

- Compactness for a general metric space
- Extra properties of $\mathbb{R}$, w.r.t. Euclidean distance
  - Right- and left-continuity
  - Lim sup and lim inf

Last time

Definition

Let (X, d) be a metric space. A subset $A \subseteq X$ is called *dense* if $\overline{A} = X$.

Definition

A metric space (X, d) is *separable* if it contains a countable dense subset.

Example

$\mathbb{R}$ is separable because $\mathbb{Q}$ is dense in $\mathbb{R}$

$\mathbb{Q}^n \subseteq \mathbb{R}^n$, so $\mathbb{R}^n$ is also separable

Example

Define $\ell_\infty = \{(x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{R},\ \sup_{n\in\mathbb{N}} |x_n| < \infty\}$, the space of bounded real valued sequences. Endow ℓ_∞ with a metric induced by the supremum norm, namely $d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sup_{n\in\mathbb{N}} |x_n - y_n|$. Then ℓ_∞ is **not separable** with respect to the topology induced by this metric.

$$\begin{cases} S \text{ countable}, & S \text{ cannot be dense} \\ S \text{ dense}, & S \text{ cannot be countable} \end{cases}$$

Compactness

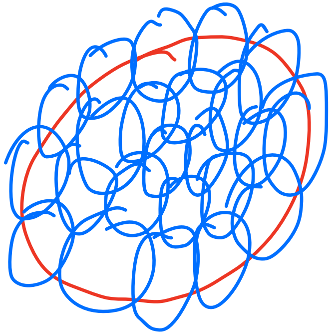
Definition

Let (X, d) be a metric space and $K \subseteq X$.

① open   ② cover

A collection $\{U_i\}_{i \in I}$ of open sets is called *open cover* of K if $K \subseteq \cup_{i \in I} U_i$.

The set K is called *compact* if for all open covers $\{U_i\}_{i \in I}$ there exists a finite subcover, meaning there exists an $n \in \mathbb{N}$ and $\{U_1, \ldots, U_n\} \subseteq \{U_i\}_{i \in I}$ such that $K \subseteq \cup_{i=1}^{n} U_i$.



K.

$\hookrightarrow U_i$ - open set

$\cup U_i \supseteq K$

Example

Let $S \subseteq X$ where (X, d) is a metric space. If S is finite, then it is compact.

Proof.

Finite set is always compact.

S is finite, so we can write it as $S = \{x_1, \cdots x_n\}$

Let $\{U_\lambda\}_{\lambda \in \Lambda}$ is an open cover of S, s.t. U_λ is open set and $S \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda$

For each i, $\exists \lambda_i \in \Lambda$, s.t. $x_i \in U_{\lambda_i}$

Then, $S = \bigcup_{i=1}^{n} \{x_i\} \underset{\uparrow}{\subseteq} \bigcup_{i=1}^{n} U_{\lambda_i}$ $\rightarrow$. S is controled by a finite sub cover.

$(\{x_i\} \subseteq U_{\lambda_i})$

Therefore, S is compact

We need to find an open cover whose finite subcover cannot cover $(0,1)$

Example

$(0,1) \subseteq \mathbb{R}$ is not compact.

Proof. Let $U_n = [\frac{1}{n}, 1)$, $n \geq 2$, $\{U_n\}_{n\geq 2}$ is an open cover of $(0,1)$

$$\begin{cases} U_n \text{ is open } (n \geq 2) \\ \bigcup_{n=1}^{\infty} U_n = (0,1) \subseteq (0,1) \end{cases}$$

(prove by contradiction)

Suppose $(0,1)$ is compact, then there is a finite subcover.

$\{U_{n_1}, U_{n_2}, \cdots, U_{n_k}\}$, $n_1 < n_2 < \cdots < n_k$, n_k is finite number.

Then, $(0,1) \subseteq \bigcup_{j=1}^{k} U_{n_j} = U_{n_k} = [1/n_k, 1) \subsetneq (0,1)$ Contradiction!

So we must have $(0,1)$ is not compact.

Proposition

Let (X, d) be a metric space and take a non-empty subset $K \subseteq X$. The following holds:

1. If X is compact and K is closed, then K is compact (i.e. closed subsets of compact sets are compact).
2. If K is compact, then K is closed. $\iff$ If k is not closed, then k is never compact.

$(0,1)$ is not compact

compact $\longrightarrow$ closed.

compact $\not\longleftarrow$ closed

If X is compact

we need show $\forall$ collection of open cover $\exists$ finite sub cover.

Proof. (1) If X is compact and $K \subseteq X$ is closed, then K is compact

Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover of K. i.e. $K \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$

since K is closed, K^c is open.

Then $\{U_\lambda\}_{\lambda\in\Lambda} \cup K^c$ is open cover of X $\left(\{U_\lambda\}_{\lambda\in\Lambda} \cup K^c \supseteq K \cup K^c = X \right)$

Since X is compact, there exists a finite sub cover of X.

1) If this ~~this~~ sub cover does ~~not~~ include K^c,
~~then~~ it must be a finite sub cover of K frome $\{U_\lambda\}_{\lambda\in\Lambda}$

2) If K^c is included in the finite sub cover. let the rest be $\{U_{\lambda_i}\}_{i=1,\cdots,n}$

Then $X = K \cup K^c \subseteq \bigcup_{i=1}^{n} U_{\lambda_i} \cup K^c$.

And since $K \cap K^c = \phi$, so we must have $K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$

K is controled by a finite sub cover.

(2) $K \subseteq X$ compact $\Rightarrow$ K is closed. $\begin{cases} \text{prove it by definition} \\ \text{sufficient condition.} \leftarrow \end{cases}$

It suffices to show $\bar{K} = K$. (Show this by contradiction)

Suppose $\bar{K} \setminus K \neq \phi$. Let $x \in \bar{K} \setminus K$

Now consider $\overline{B_\varepsilon(x)}^c$, $\varepsilon > 0$.

① $\overline{B_\varepsilon(x)}^c$ is open

② $\bigcup_{\varepsilon > 0} \overline{B_\varepsilon(x)}^c = X \setminus \{x\}$

Since $x \in \bar{K} \setminus K$, $x \notin K$. So we have $K \subseteq \bigcup_{\varepsilon > 0} \overline{B_\varepsilon(x)}^c = X \setminus \{x\}$

Thus, $\{\overline{B_\varepsilon(x)}^c\}_{\varepsilon > 0}$ is an open cover of K.

Since K is compact, there $\exists$ a finite subcover.

Let this subcover be $\{\overline{B_{\varepsilon_1}(x)}^c, \overline{B_{\varepsilon_2}(x)}^c, \cdots, \overline{B_{\varepsilon_n}(x)}^c\}$

rank it in increasing order

$\varepsilon_1 > \varepsilon_2 > \varepsilon_3 \cdots > \varepsilon_n$

Then $K \subseteq \overline{B_{\varepsilon_n}(x)}^c$ $\left(\overline{B_{\varepsilon_i}(x)}^c \subset \overline{B_{\varepsilon_j}(x)}^c, \forall i < j \right)$.

$\overline{B_{\varepsilon_n}(x)} \subseteq K^c$, Therefore $B_{\varepsilon_n}(x) \cap K = \phi$ $(K^c \cap K = \phi)$

contradiction!

However, we've assumed $x \in \overline{K}$, $B_{\varepsilon_n}(x) \cap K \neq \phi$

So we must have, for any compact set K,

$\overline{K} \setminus K$ is empty set.

That saying, K is closed.

Arbitrary compact metric spaces have some nice properties in general as the next proposition shows.

Any Cauchy sequences must be convergent sequence

Proposition

$\exists$ dense and countable subset

A compact metric space (X, d) is complete and separable.

Also, just as we had a sequential characterization of the closure of a set in metric spaces, we similarly have a sequential characterization of compactness.

Theorem

Let (X, d) be a metric space. Then $K \subseteq X$ is compact with respect to the metric induced by d if and only if every sequence in K admits a subsequence converging to some point in K.

Compactness on $\mathbb{R}^n$

Recall: for a general metric space.

compact $\Longrightarrow$ closed.

$\Longleftarrow$ X is compact.

Metric space ($\mathbb{R}^n$, 2-norm)

Theorem (Heine-Borel Theorem)

Let $K \subseteq \mathbb{R}^n$. Then K is compact with respect to the topology induced by the Euclidean distance if and only if it is closed and bounded.

compact $\Longleftrightarrow$ closed and bounded.

A corollary of the last two theorems is the Bolzano-Weierstrass theorem.

Corollary (Bolzano-Weierstrass)

Any bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Let $\{x_n\}$ be a bounded sequence. Then $\exists M>0$, s.t. $|x_n|<M$, $\forall n$.

Then $\{x_n\} \subset K := \{|x| \le M\}$. closed and bounded. By H-B theorem,

K is compact. $\xRightarrow{\text{def}}$ $\exists$ subsequence of $\{x_n\}$, converging to some point K.

The compactness remains under cts function.

Proposition

Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose $K \subseteq X$ is compact and let $f\colon K \to Y$ be continuous. Then $f(K)$ is compact. If f is cts, then image of compact set is still compact

Note: this is a generalization of the Extreme Value Theorem to metric spaces.

$f: [a,b] \to \mathbb{R}$, cts $\Rightarrow f$ has min and max.

$:= K$, compact. f is cts $\Rightarrow f(K)$ is compact (closed and bounded)

$\Rightarrow f(K)$ has min and max.

Recall from the set theory section:
If $f : X \to Y$:

1. $A \subseteq B \subseteq Y \Rightarrow f^{-1}(A) \subseteq f^{-1}(B)$ and $A \subseteq B \subseteq X \Rightarrow f(A) \subseteq f(B)$
2. $f^{-1}(\cup_{i \in I} A_i) = \cup_{i \in I} f^{-1}(A_i)$, where $A_i \subseteq Y \forall i \in I$
3. $f(\cup_{i \in I} A_i) = \cup_{i \in I} f(A_i)$, where $A_i \subseteq X \forall i \in I$
4. $A \subseteq X \Rightarrow A \subseteq f^{-1}(f(A))$ But we do not $A \supseteq f^{-1}(f(A))$ in general.
5. $B \subseteq Y \Rightarrow f(f^{-1}(B)) \subseteq B$

Proof. Assume f is cts. Let $K \subseteq X$ be any compact set.
we will prove $f(K)$ is compact by definition.

Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover $f(K)$, i.e. $f(K) \subseteq \bigcup_{\lambda\in\Lambda} U_\lambda$

Then
$$K \subseteq f^{-1}(f(K)) \quad \text{④ in previous page}$$
$$\subseteq f^{-1}\Big(\bigcup_{\lambda\in\Lambda} U_\lambda\Big)$$
$$= \bigcup_{\lambda\in\Lambda} f^{-1}(U_\lambda) \quad \text{② in previous page.}$$

This means $\{f^{-1}(U_\lambda)\}_{\lambda\in\Lambda}$ is an open cover of K.

Since K is compact, $\exists$ finite subcover of K.

Let it be $\{ f^{-1}(U_{\lambda_i}) \}_{i=1,2,\cdots,n}$, and we have

$$K \subseteq \bigcup_{i=1}^{n} f^{-1}(U_{\lambda_i})$$

$$f(K) \subseteq f\left(\bigcup_{i=1}^{n} f^{-1}(U_{\lambda_i})\right)$$

$$= \bigcup_{i=1}^{n} f\left(f^{-1}(U_{\lambda_i})\right)$$ 3 in previous page.

$$\subseteq \bigcup_{i=1}^{n} U_{\lambda_i}$$ $\left(f(f^{-1}(U_{\lambda_i})) \subseteq U_{\lambda_i} \right.$ 5 in previous page.

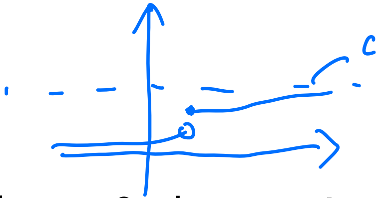
Thus, $\{U_{\lambda_i}\}_{i=1}^{n}$ is finite open cover of $f(K)$.

subcover.

That's saying $f(K)$ is also compact.

Extra properties of $\mathbb{R}$

Right and left continuous

E.g. cdf  cdf of a r.v. X

right cts.

Recall: $f\colon \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|x_0 - y| < \delta$ implies $|f(x_0) - f(y)| < \epsilon$.

$$\lim_{x \to x_0^-} f(x) = f(x_0)$$

$\Updownarrow$

Definition

Let $f\colon \mathbb{R} \to \mathbb{R}$.

- f is *left continuous* at $x_0 \in \mathbb{R}$ if for all $\epsilon > 0$ there exists a $\delta > 0$, such $|f(x_0) - f(x)| < \epsilon$ whenever $x_0 - \delta < x < x_0$.
- f is *right continuous* at $x_0 \in \mathbb{R}$ if for all $\epsilon > 0$ there exists a $\delta > 0$, such $|f(x_0) - f(x)| < \epsilon$ whenever $x_0 < x < x_0 + \delta$.

We say that f is left continuous if it is left continuous at all points in the domain, and similar for right continuous.

Proposition

A function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous if and only if it is left and right continuous.

Proof. Exercise.

Bounded sequences and monotone convergence

Definition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. We call $(x_n)_{n\in\mathbb{N}}$ *bounded* if there exists an $M > 0$ such that $|x_n| < M$ for all $n \in \mathbb{N}$.

Theorem (Monotone convergence theorem)

(i) *Suppose $(x_n)_{n\in\mathbb{N}}$ is an increasing sequence, i.e. $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$, and that it is bounded (above). Then the sequence* 1. *converges. Furthermore,*
2. $\lim_{n\to\infty} x_n = \sup_{n\in\mathbb{N}} x_n$, *where* $\sup_{n\in\mathbb{N}} x_n := \sup\{x_n : n \in \mathbb{N}\}$.

(ii) *Suppose $(x_n)_{n\in\mathbb{N}}$ is a decreasing sequence, i.e. $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$, which is bounded (below). Then the sequence converges and* $\lim_{n\to\infty} x_n = \inf_{n\in\mathbb{N}} x_n := \inf\{x_n : n \in \mathbb{N}\}$.

Convention: $\sup A = \infty$ if $A \subseteq \mathbb{R}$ is not bounded above and $\inf A = -\infty$ if A is not bounded below.

Lemma

If $A \subseteq B \subseteq \mathbb{R}$ is non-empty, then $\inf A \leq \sup A$, $\sup A \leq \sup B$, and $\inf A \geq \inf B$.

$\inf B \leq \sup B$

The proof of this follows from the definition of greatest lower and least upper bound.

Definition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. We define the *limit superior* of $(x_n)_{n\in\mathbb{N}}$ as

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} \boxed{\sup_{k\geq n} x_k}.$$

← a sequence induced by n. decreases as n increases

Similarly we define the *limit inferior* of $(x_n)_{n\in\mathbb{N}}$ as

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} \boxed{\inf_{k\geq n} x_k}.$$

← a sequence induced by n increasing.

If the sequence $(x_n)_{n\in\mathbb{N}}$ is not bounded above, then $\limsup_{n\to\infty} x_n = \infty$. Similarly, if the sequence $(x_n)_{n\in\mathbb{N}}$ is not bounded below, then $\liminf_{n\to\infty} x_n = -\infty$.

Proposition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$.

- The sequence of suprema, $s_n = \sup_{k\geq n} x_k$, is decreasing and the sequence of infima, $i_n = \inf_{k\geq n} x_k$, is increasing.
- The limit superior and the limit inferior of a bounded sequence always exist and are finite.

Proof.

exercise.

By MCT. For a bounded sequence, if it is monotone, the it's convergent $\iff$ limit exists.

For a general bounded sequence, limit is not guaranteed to exist.

$\triangle$. For a general bounded sequence, limsup and liminf always exist.

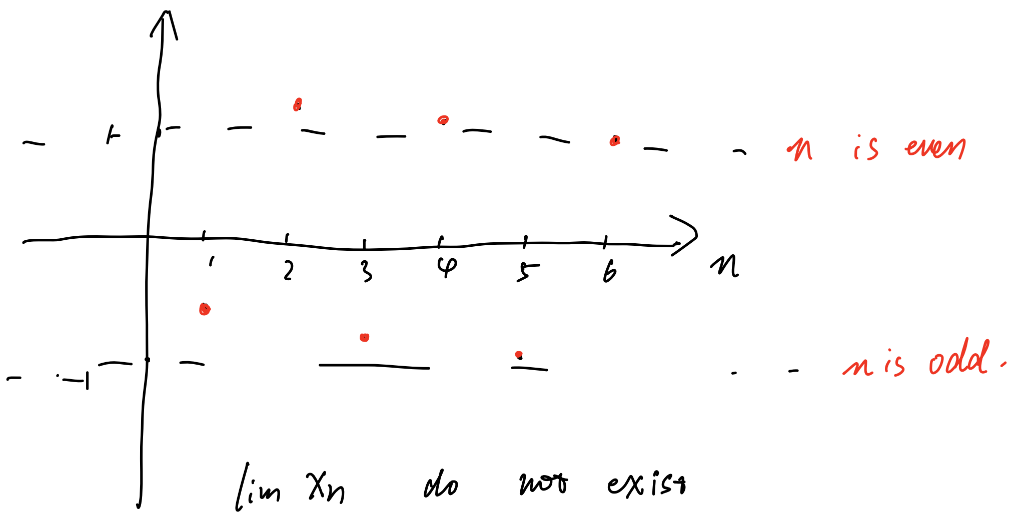
Theorem

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Then the sequence converges to $x \in \mathbb{R}$ if and only if $\limsup_{n\to\infty} x_n = x = \liminf_{n\to\infty} x_n$.

Proof in notes.

$\lim_{n\to\infty} x_n = x$ iff $\limsup_{n\to\infty} x_n = x = \liminf_{n\to\infty} x_n$.

E.g. $x_n = (-1)^n + (1/2)^n$



$\lim_{n\to\infty} x_n$ do not exist

$\limsup_{n\to\infty} x_n = 1$

$\liminf_{n\to\infty} x_n = -1$

We can extend this easily to a sequence of functions $f_n \colon X \to \mathbb{R}$ as follows:

Define $f = \limsup_{n\to\infty} f_n$ to be the function defined pointwise by $f(x) = \limsup_{n\to\infty}(f_n(x))$ and similar for the limit inferior.

$x \in X$

There also exists a set theoretic version in terms of unions and intersections which you will encounter in probability.

for limsup, liminf

References

Runde, Volker (2005). *A Taste of Topology*. Universitext. url: https://link.springer.com/book/10.1007/0-387-28387-0