Module 7: Linear Algebra I

Operational math bootcamp

Statistical Sciences
UNIVERSITY OF TORONTO

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Outline

Today:

- Vector spaces and subspaces
- Linear independence and bases
- Linear maps, null space, range

Vector spaces & subspaces

Closure under addition and multiplication. Next Mon.

$\mathbb{C}$ or $\mathbb{R}$

Already assumed $(+): V \times V \to V$ $(\times): \mathbb{F} \times V \to V$

Definition

We call V a **vector space** if the following hold:

(A) *Commutativity in addition:* $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in V$

(B) *Associativity in addition:* $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$

(C) *Existence of a neutral element, addition:* There exists a vector $\mathbf{0}$ such that for any $\mathbf{v} \in V$, $\mathbf{0} + \mathbf{v} = \mathbf{v}$

$\neq 0 \in \mathbb{R}$

(D) *Additive inverse:* For every $\mathbf{v} \in V$, there exists another vector, which we denote $-\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.

(E) *Existence of a neutral element, multiplication:* For any $\mathbf{v} \in V$, $1 \times \mathbf{v} = \mathbf{v}$

(F) *Associativity in multiplication:* Let $\alpha, \beta \in \mathbb{F}$. For any $\mathbf{v} \in V$, $(\alpha\beta)\mathbf{v} = \alpha(\beta\mathbf{v})$

(G) Let $\alpha \in \mathbb{F}, \mathbf{u}, \mathbf{v} \in V$. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$.

(H) Let $\alpha, \beta \in \mathbb{F}, \mathbf{v} \in V$. $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$.

Notation used in notes $\begin{cases} \text{scalar}: \alpha, \beta, 0 \\ \text{Vector}: \vec{u}, \vec{v}, \vec{0} \end{cases}$

Elements of the vector space are called vectors.
Most often we will assume $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$.

Once we define a appropriate (+) and (X).

Example

The following are vector spaces:

- $\mathbb{R}^n$
- $\mathbb{C}^n$
- $C(\mathbb{R};\mathbb{R})$, continuous functions from $\mathbb{R}$ to $\mathbb{R}$ o.
- $M_{n\times m}$, matrices of size $n \times m$
- $\mathbb{P}_n$ (polynomials of degree n, $p(x) = a_0 + a_1x + \ldots + a_nx^n$).

$\alpha \in \mathbb{R}$, $f, g \in C(\mathbb{R}, \mathbb{R})$

$\Rightarrow f+g \in C(\mathbb{R}, \mathbb{R})$

$\alpha f \in C(\mathbb{R}, \mathbb{R})$

o. (polynomial) + (polynomial) = (polynomial)

α (polynomial) = (polynomial).

scalar vector.

Lemma

For every $\mathbf{v} \in V$, $0\mathbf{v} = \mathbf{0}$.

Proof.

Since $\underbrace{0 = 0 + 0}_{\text{scalar,}}$ by (H1) $0 \cdot \vec{v} = (0+0)\,\vec{v} \overset{(H1)}{=} 0\vec{v} + 0\vec{v}$

By (D), there is always an additive inverse of $0 \cdot \vec{v}$

So. $\vec{0} = 0 \cdot \vec{v} + (-0 \cdot \vec{v}) = 0\vec{v} + \underbrace{0\vec{v} + (-0 \cdot \vec{v})}_{\vec{0}}.$

$= 0\vec{v} + \vec{0}$

$\Rightarrow \vec{0} = 0\vec{v}$

$\square$

Lemma

For every $\mathbf{v} \in V$, we have $-\mathbf{v} = (-1) \times \mathbf{v}$.

Def of $-\vec{v}$: $\vec{v} + (-\vec{v}) = \vec{0}$

Proof.

We need to show $\vec{v} + (-1) \times \vec{v} = \vec{0}$ for any $\vec{v} \in V$

First we have $\underbrace{0 = 1 + (-1)}_{\text{sclar}}$, $\underbrace{0 \cdot \vec{v}}_{= \vec{0} \text{ by previous page.}} = \underbrace{1 \cdot \vec{v}}_{= \vec{v} \text{ by (G) } 1 \cdot \vec{v} = \vec{v}} + (-1)\vec{v}$

$\Rightarrow \vec{0} = \vec{v} + (-1) \cdot \vec{v}$

That's definition of $-\vec{v}$. //

□

Definition

A subset U of V is called a **subspace** of of V if U is also a vector space (using the same addition and scalar multiplication as on V).

Proposition

A subset U of V is a subspace of V if and only if U satisfies the following three conditions:

vector.

1. $\mathbf{0} \in U$
2. Closed under addition: $\mathbf{u}, \mathbf{v} \in U$ implies $\mathbf{u} + \mathbf{v} \in U$
3. Closed under scalar multiplication: $\alpha \in \mathbb{F}$ and $\mathbf{u} \in U$ implies $\alpha\mathbf{u} \in U$

$\forall \alpha \in \mathbb{F}$ and $\vec{u}, \vec{v} \in U$,

we have $\alpha(\vec{u} + \vec{v}) \in U$

②③ holds trivially since U is a vector space.

Proof. ($\Rightarrow$) Need to show $\vec{0}_V \in U$

Let $\vec{u} \in U$. Then $(-1)\cdot\vec{u} \in U$ (by ③)

$$\vec{u} + (-1)\cdot\vec{u} = 1\cdot\vec{u} + (-1)\cdot\vec{u} = (1+(-1))\cdot\vec{u} = 0\cdot\vec{u} = \vec{0}_V$$

by ②, we have $\vec{u} + (-1)\cdot\vec{u} \in U$

so $\vec{0}_V \in U$

($\Leftarrow$) Need to show (D) and (E), the existence of additive inverse

(E) holds by ②, $\vec{0} \in U$

For any $\vec{u} \in U \subseteq V$, $\exists -\vec{u} = (-1)\cdot\vec{u} \in V$

by ③ $(-1)\cdot\vec{u} \in U$

$\Rightarrow (-1)\cdot\vec{u} = -\vec{u} \in U$

That's saying (D) and (E) hold.

So U is also a

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(C) *Existence of a neutral element, addition:* There exists a vector $\mathbf{0}$ such that for any $\mathbf{v} \in V$, $\mathbf{0} + \mathbf{v} = \mathbf{v}$ $\neq 0 \in \mathbb{R}$

(D) *Additive inverse:* For every $\mathbf{v} \in V$, there exists another vector, which we denote $-\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.

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(G) Let $\alpha \in \mathbb{F}, \mathbf{u}, \mathbf{v} \in V$. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$.

(H) Let $\alpha, \beta \in \mathbb{F}, \mathbf{v} \in V$. $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$.

A vector space should be closed under addition and scalar multiplication.

(A) — addition

(B) — addition

(C) — vector space V, $\vec{0}_V \in V$

(D) — vector space V, $-\vec{v} \in V$

(E) — multiplication

(F) — multiplication

(G) — addition and multiplication

(H) —

vector space.

Linear (in)dependence and bases

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Linear combinations

Definition

A linear combination of vectors $\mathbf{v}_1, ..., \mathbf{v}_n$ of vectors in V is a vector of the form

$$\alpha_1 \mathbf{v}_1 + ... + \alpha_n \mathbf{v}_n = \sum_{k=1}^{n} \alpha_k \mathbf{v}_k$$

weighted sum

where $\alpha_1, ..., \alpha_m \in \mathbb{F}$.

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Span

Definition

The set of all linear combinations of a list of vectors $\mathbf{v}_1, ..., \mathbf{v}_n$ in V is called the **span** of $\mathbf{v}_1, ..., \mathbf{v}_n$, denoted span$\{\mathbf{v}_1, ..., \mathbf{v}_n\}$. In other words,

$$\text{span}\{\mathbf{v}_1, ..., \mathbf{v}_n\} = \{\alpha_1 \mathbf{v}_1 + ... + \alpha_m \mathbf{v}_n : \alpha_1, ..., \alpha_n \in \mathbb{F}\}$$

The span of the empty list is defined to be $\{\mathbf{0}\}$.

Naturally, we have span$\{\vec{v}_1, \cdots, \vec{v}_n\}$ is subspace of V.

① linear combination $\in V$

②. span$\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a vector space.

once we can show (D) and (G) hold.

Basis

Definition

A system of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is called a basis (for the vector space V) if any vector $\mathbf{v} \in V$ admits a unique representation as a linear combination

there is only one linear combination.

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \ldots + \alpha_n \mathbf{v}_n = \sum_{k=1}^{n} \alpha_k \mathbf{v}_k. \in \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_n\}$$

Example

$\mathbb{F}$ can be $\mathbb{R}$, $\mathbb{C}$

- For $\mathbb{F}^n$, $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $e_n = (0, \ldots, 0, 1)$ is a basis
- The monomials $1, x, x^2, \ldots, x^n$ form a basis for $\mathbb{P}_n$. polynomial functions of degree n.

$\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_n \cdot x^n$

Linear independence

If $v_1, \dots, v_n$ are linear dependent

$\exists\, \alpha_i \neq 0$, s.t. $\sum_{i=1}^{n} \alpha_i \cdot \vec{v_i} = \vec{0} \Rightarrow \alpha_i \cdot \vec{v_i} = -\sum_{j \neq i} \alpha_j \vec{v_j}$

Since $\alpha_i \neq 0$, $\vec{v_i} = -\sum_{j \neq i} \frac{\alpha_j}{\alpha_i} \vec{v_j}$ $\Rightarrow$ v_i is a linear combination of $\{v_i\}$ except v_i

Definition

A system of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ in V is called *linearly independent* if

$$\sum_{i=1}^{n} \alpha_i \mathbf{v}_i = \mathbf{0}$$

$\exists!\ \alpha$, s.t. $\alpha_1 = \alpha_2 = \cdots \alpha_n = 0$

implies $\alpha_i = 0$ for all $i = 1, \dots, n$.

Otherwise, we call the system *linearly dependent*.

Linear combinations $\alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n$ such that $\alpha_k = 0$ for every k are called trivial.

Spanning set

Definition

A system of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in V is called *spanning* if any vector in V can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$. In other words,

$$V = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}.$$

Such a system is also often called generating or complete. The next proposition relates spanning and linearly independent to a basis.

① existence of linar combination

② uniqness of such representation.

Proposition

A system of vectors $\mathbf{v}_1, \ldots \mathbf{v}_n \in V$ is a basis if and only if it is linearly independent and spanning.

Proof. ($\Rightarrow$) $\forall \vec{v} \in V$, $\vec{v}$ can be written as a linear combination of $\vec{v}_1, \cdots \vec{v}_n$ and there is only one representation.

$\Updownarrow$

$V = \text{span}\{\vec{v}_1, \vec{v}_2, \cdots \vec{v}_n\}$

So it suffices to show $\vec{v}_1, \cdots \vec{v}_n$ are linearly independent.

Let $\sum_{i=1}^{n} \alpha_i \cdot \vec{v}_i = \vec{0}$, since $\alpha_i = 0, \forall i$ is a solution,

by the uniqueness of such representation ($\vec{0} \in V$, $\vec{0}$ can be written as a linear combn of $\vec{v}_1, \cdots \vec{v}_n$)

we must have $\alpha_i = 0, \forall i$ is unique solution.

($\Leftarrow$) suppose $V = \text{span}\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\rangle$ and $\{\vec{v}_1, \dots \vec{v}_n\}$ are linear independent

It suffices to show uniqueness of linear representation by $\{\vec{v}_1, \dots \vec{v}_n\}$.

Assume $\vec{v} = \sum_{i=1}^{n} \alpha_i \vec{v}_i = \sum_{i=1}^{n} \beta_i \vec{v}_i \Rightarrow \vec{0} = \vec{v} - \vec{v} = \sum_{i=1}^{n} (\alpha_i - \beta_i) \cdot \vec{v}_i$

Since $\{\vec{v}_1, \dots \vec{v}_n\}$ are linear independent, $\alpha_i - \beta_i = 0, \forall i$

$\Leftrightarrow \alpha_i = \beta_i$

$\Leftrightarrow$ linear representation is unique

Proposition

Let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ be spanning. Then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ contains a basis.

Sketch of proof.

$\begin{cases} \text{Spanning: } V = \operatorname{span}\{\vec{v}_1, \ldots \vec{v}_n\} \\ \text{span: the set of all linear combination} \end{cases}$

Define $E_1 = \{\vec{v}_1\}$.

If $\begin{cases} \vec{v}_2 \in \operatorname{span}\{\vec{v}_1\}, \text{ then let } E_2 = \{\vec{v}_1\} \\ \vec{v}_2 \notin \operatorname{span}\{\vec{v}_1\}, \text{ then Let } E_2 = \{\vec{v}_1, \vec{v}_2\} \end{cases}$

$\vdots$

If $\vec{v}_i \begin{cases} \in E_{i-1}, \text{ then let } E_i = E_{i-1} \\ \notin E_{i-1}, \text{ then let } E_i = E_{i-1} \cup \{\vec{v}_i\} \end{cases}$

Repeat this process, we have E_n spanning V
and E_n are linear independent

Definition

An $\mathbb{F}$-vector space V is called *finite dimensional* if there exists a finite list of vectors that span it, i.e. there exist $n \in \mathbb{N}$ and $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ such that $V = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Otherwise, we call V *infinite dimensional*.

Example

$\mathbb{C}^n$ or $\mathbb{R}^n$

- $\mathbb{F}^n$, $M_{m\times n}$, $\mathbb{P}_n$ are examples of finite dimensional vector spaces
- The $\mathbb{F}$-vector space $\mathbb{P} = \{\sum_{i=1}^{n} \alpha_i x^i \ : \ n \in \mathbb{N}, \alpha_i \in \mathbb{F}, i = 1, \ldots, n\}$ is infinite dimensional.

$\neq \mathbb{P}_n$

Corollary

Every finite dimensional vector space has a basis.

This follows from the fact that every spanning set for a vector space contains a basis.

$\Rightarrow$ have a basis.

This can also be extended to infinite dimensional vector spaces, i.e. when we do not assume that there exists a finite spanning set. However, this relies on the Axiom of Choice and is beyond the scope of this course.

e.g. we can take $\{1, x, x^2, \dots x^n, \dots\}$ as a basis of $\mathbb{P}$

Proposition

Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

Proof. Let $\vec{u}_1, \cdots \vec{u}_n$ be linearly independent

Let $\vec{v}_1, \cdots \vec{v}_m$ be a basis of V.

Set $E_0 = \{\vec{u}_1, \cdots, \vec{u}_n\}$.

Procedure: If $\vec{v}_1 \notin \operatorname{span} E_0$, then $E_1 = E_0 \cup \{\vec{v}_1\}$

$\in \operatorname{span} E_0$, then $E_1 = E_0$

Repeat this to $\vec{v}_m$ and get E_m

we can show E_m is a basis of V.

Dimension

Proposition

Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and $\mathbf{u}_1, \ldots, \mathbf{u}_m$ be a basis for V. Then $m = n$.

The proof is omitted,. It relies on the fact that the number of elements in linearly independent systems are always less than or equal to the number of elements in spanning systems. $m \leq n, \quad n \leq m \Rightarrow m = n.$

Definition

Let V be a finite dimensional $\mathbb{F}$-vector space. The number of elements in a basis of V is called the *dimension* of V and is denoted $\dim(V)$.

By the previous definition, the notion of dimension is well-defined.

for a vector space V, there is only one dimension.

Dimension

Example

- $\dim(\mathbb{F}^n) = n$
- $\dim(\mathbb{P}_n) = n+1$ $\{1, x, x^2, \cdots, x^n\}$ – $n+1$ elements
- $\dim\{\mathbf{0}\} = 0$ there is no basis

Linear maps

Linear Maps

$T: U \to V$, $T(\vec{u})$, $T(\vec{v}) \in V$

Definition

A map from a vector space U to a vector space V is **linear** if

T

$$T(\alpha \mathbf{u} + \beta \mathbf{v}) = \alpha T(\mathbf{u}) + \beta T(\mathbf{v}) \quad \text{for any } \mathbf{u}, \mathbf{v} \in \overset{U}{\cancel{V}},\ \alpha, \beta \in \mathbb{F}$$

Notation: $\mathcal{L}(U, V)$ is the set of all linear maps from $\mathbb{F}$-vector space U to $\mathbb{F}$-vector space V

Example

- Zero map $0: U \to V$ defined by $0(\vec{u}) = \vec{0} \in V$ for $\vec{u} \in U$
- Identity map $Id: V \to V$ defined by $Id(\vec{v}) = \vec{v}$ for $\vec{v} \in V$
- Differentiation $D: P(\mathbb{R}) \to P(\mathbb{R})$

$$\begin{array}{ccc} P(\mathbb{R}) & \to & P(\mathbb{R}) \\ \downarrow & & \downarrow \\ x^n & \longrightarrow & n \cdot x^{n-1} \end{array}$$

$\dim(U) = \dim(V)$

Theorem

Suppose $\mathbf{u}_1, \ldots, \mathbf{u}_n$ is a basis for U and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis for V. Then there exists a unique linear map $T : U \to V$ such that $T\mathbf{u}_j = \mathbf{v}_j$ for $j = 1, \ldots, n$.

Proof in book.

The set of linear map is a vector space.

Theorem

Let $S, T \in \mathcal{L}(U, V)$ and $\alpha \in \mathbb{F}$. $\mathcal{L}(U, V)$ is a vector space with addition defined as the sum $S + T$ and multiplication as the product αT.

The proof follows from properties of linear maps and vector spaces. Note that the additive identity is the zero map.

zero vector

$$(S+T)(u) \overset{\text{def}}{=} S(u) + T(u)$$

$$(\alpha T)(u) \overset{\text{def}}{=} \alpha T(u)$$

$\vec{0}_u \in U$ $\vec{0}_v \in V$

Lemma

Let $T \in \mathcal{L}(U, V)$. Then $T(\mathbf{0}) = \mathbf{0}$.

Proof. Since $\vec{0}_u = \vec{0}_u + \vec{0}_u$, by linearity of T

$$T(\vec{0}_u) = T(\vec{0}_u + \vec{0}_u) = T(\vec{0}_u) + T(\vec{0}_u)$$

Also, we have $T(\vec{0}_u) \in V$

$$\Rightarrow T(\vec{0}_u) = \vec{0}_v$$

Null space and range

Definition

Let $T : U \to V$ be a linear transformation. We define the following important subspaces:

ker T $\qquad$ $\vec{0}_V \in V$

- *Kernel or null space*: null $T = \{\mathbf{u} \in U : T\mathbf{u} = \mathbf{0}\}$
- *Range*: range $T = \{\mathbf{v} \in V : \exists \mathbf{u} \in U \text{ such that } \mathbf{v} = T\mathbf{u}\}$ ~ "image of U under map T"

The dimensions of these spaces are often called the following:

- *Nullity*: nullity$(T) = \dim(\text{null}(T))$
- *Rank*: rank$(T) = \dim(\text{range}(T))$

If $U \equiv V$, then

$$\text{rank}(T) + \text{nullity}(T) = \dim(\text{range}(T)) + \dim(\ker(T)) = \dim(U)$$

Proposition

Let $T: U \to V$. The null space of T is a subspace of U and the range of T is a subspace of V.

Proof. (i) the null space of T is a subspace of U

a) $\vec{0}_u \in \ker T$, it is already proved by previous lemma s.t. $T(\vec{0}_u) = \vec{0}_v$

b) closed under addition

Let $\vec{u}, \vec{v} \in \ker T$, then $T\vec{u} = T\vec{v} = \vec{0}_v$

Thus $T(\vec{u}+\vec{v}) = T\vec{u} + T\vec{v} = \vec{0}_v + \vec{0}_v = \vec{0}_v$

$\Rightarrow \vec{u}+\vec{v} \in \ker T$

c) closed under scalar multiplication.

Let $\alpha \in \mathbb{F}$, $\vec{v} \in \ker T$

Then $T(\alpha \vec{v}) = \alpha \cdot T(\vec{v}) = \alpha \cdot \vec{0}_v = \vec{0}_v \Rightarrow \alpha \cdot \vec{v} \in \ker(T)$

By a) ~ c), we conclude $\ker T$ is a subspace of U

(ii) the range of T is a subspace of V

By definition, we have range(T) $\subseteq V$

a) By previous lemma, $(T(\vec{0}_U) = \vec{0}_V)$. So $\vec{0}_V \in$ range(T)

b) closed-addition Let $\vec{v}_1, \vec{v}_2 \in$ range(T), then there exist $\vec{u}_1, \vec{u}_2$ s.t. $\vec{v}_1 = T(\vec{u}_1)$, $\vec{v}_2 = T(\vec{u}_2)$

Thus $\vec{v}_1 + \vec{v}_2 = T(\vec{u}_1) + T(\vec{u}_2) = T(\vec{u}_1 + \vec{u}_2) \in$ range(T)

c) closed-multiplication

Let $\alpha \in \mathbb{F}$, $\vec{v} \in$ range(T)

$\exists\ \vec{u} \in U$, s.t. $\vec{v} = T(\vec{u})$

$\alpha\vec{v} = \alpha T(\vec{u}) = T(\underbrace{\alpha\vec{u}}_{\in U}) \in$ range(T)

Therefore ...

Not a coincidence.

Example

Zero map from a vector space U to a vector space V:

- The null space is U
- The range is $\{\vec{0}_V\}$.

If $\dim(U)$ is finite,

$\dim(\ker T) + \dim(\operatorname{range} T)$

$= \dim(U) + \dim(\{\vec{0}_V\})$

$= \dim(U)$

Differentiation map from $\mathbb{P}(\mathbb{R})$ to $\mathbb{P}(\mathbb{R})$:

- The null space is $\mathbb{P}_0(\mathbb{R})$ or the set of all constants.
- The range is $\mathbb{P}(\mathbb{R})$

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$\vec{u} \neq \vec{v} \Rightarrow T(\vec{u}) \neq T(\vec{v})$

$\Updownarrow$

Definition (Injective and surjective)

Let $T : U \to V$. T is *injective* if $T\mathbf{u} = T\mathbf{v}$ implies $\mathbf{u} = \mathbf{v}$ and T is *surjective* if $\forall \mathbf{v} \in V, \exists \mathbf{u} \in U$ such that $\mathbf{v} = T\mathbf{u}$, i.e. if range $T = V$.

Theorem

$T \in \mathcal{L}(U, V)$ *is injective if and only if* null $T = \{\mathbf{0}\}$.

$\ker T = \{\vec{0}_u\}$

$\Leftrightarrow$ T maps all non-zero vector into non-zero vector $\Leftrightarrow$ T is injective.

Proof. ($\Rightarrow$) Suppose T is injective.

Let $\vec{v} \in \ker T$, Then $T\vec{u} = \vec{0}_V = T \cdot \vec{0}_U$

Since T is injective, $\vec{u} = \vec{0}_U$, which means $\ker T = \{\vec{0}_U\}$

($\Leftarrow$) Suppose $\ker T = \{\vec{0}_U\}$

Let $\vec{u}_1, \vec{u}_2 \in U$ s.t. $T\vec{u}_1 = T\vec{u}_2$

Then $T(\vec{u}_1 - \vec{u}_2) = T(\vec{u}_1) - T(\vec{u}_2) = \vec{0}_V \Rightarrow \vec{u}_1 - \vec{u}_2 \in \ker T$

But $\ker T = \{\vec{0}_U\} \Rightarrow \vec{u}_1 - \vec{u}_2 = \vec{0}_U \Rightarrow \vec{u}_1 = \vec{u}_2$ //

Theorem (Rank Nullity Theorem)

Let $T : U \to V$ be a linear transformation, where U and V are finite-dimensional vector spaces. Then

$$\text{rank } T + \text{nullity } T = \dim U.$$

$$\dim(\text{range}(T)) + \dim(\ker T) = \dim U$$

Proof in the lecture notes (pg. 35).

E.g. $O : U \to V$

$O(\vec{u}) = \vec{0}_V$, Then $\ker T = U$

$\text{range } T = \{\vec{0}_V\}$.

$\dim(\ker T) = \dim(U)$

$\dim(\text{range } T) = 0$.

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