UNIVERSITY OF TORONTO | Statistical Sciences

DoSS Summer Bootcamp Probability Module 10

Miaoshiqi (Shiki) Liu

University of Toronto

July 28, 2022

Recap

Learnt in last module:

- Markov Chain
  - ▷ Markov Property
- Discrete-time Markov Chain
  - ▷ Transition probability
  - ▷ Chapman-Kolmogorov equation
- Continuous-time Markov Chain
  - ▷ Transition probability
  - ▷ Chapman-Kolmogorov equation
  - ▷ Generator matrix

Outline

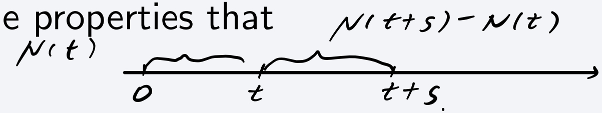
- Poisson process
  - Poisson-Gamma relationship
  - Properties of Poisson Process
- Brownian motion
  - Properties of Brownian motion
  - Brownian motion with drift
  - Geometric Brownian motion

Poisson process

Poisson process: an example of CTMC

Poisson process

A Poisson process $\{N(t)\}_{t \geq 0}$ with intensity $\lambda > 0$ is a collection of non-decreasing integer-valued random variables satisfying the properties that



- $N(0) = 0$;
- Independent increments: $N(t)$ is independent of $N(t+s) - N(t)$;
- $N(t+s) - N(s) \sim Poisson(\lambda t), \quad t \geq 0, s \geq 0.$

Poisson process

Poisson process: an example of CTMC

Poisson process

A Poisson process $\{N(t)\}_{t\geq 0}$ with intensity $\lambda > 0$ is a collection of non-decreasing integer-valued random variables satisfying the properties that

- $N(0) = 0$;
- Independent increments: $N(t)$ is independent of $N(t+s) - N(t)$;
- $N(t+s) - N(s) \sim Poisson(\lambda t), \quad t \geq 0, s \geq 0.$

$s = 0$

Remark:

- Easy to verify the Markov property of Poisson process;
- $N(t) \sim Poisson(\lambda t)$.

UNIVERSITY OF TORONTO

Poisson process

Examples:

- The number of customers arriving at a grocery store with intensity $\lambda = 5$ customers per hour; $N(2) \sim Poisson(10)$
- The number of students coming to the TA session with intensity $\lambda = 3$ students per hour;
- The number of births in Canada with intensity $\lambda = 40$ per hour.

Poisson process

Examples:

- The number of customers arriving at a grocery store with intensity $\lambda = 5$ customers per hour;
- The number of students coming to the TA session with intensity $\lambda = 3$ students per hour;
- The number of births in Canada with intensity $\lambda = 40$ per hour.

The probability that more than 60 babies are born between 9 to 11 AM in Canada:

$N(t+2) - N(t)$ $\qquad N(2) - N(0)$

$$\mathbb{P}(N(t+2) - N(t) > 60) = \mathbb{P}(\underline{N(2)} > 60) = 1 - \sum_{k=0}^{60} \frac{e^{-40\cdot 2}(40 \cdot 2)^k}{k!}$$

$t = 9$ $\qquad$ Poisson (40×2)

Poisson process

$N(t+s) - N(t) \sim \text{Poisson}(\lambda s)$ $s \geq 0$

Think about the waiting time for the event:

$T_1 \quad T_2 \quad t$

$N(t) = i$

$t := N^{-1}(i)$

Inter-arrival time for Poisson process

Consider a Poisson process $\{N(t)\}_{t\geq 0}$ with intensity λ, and let T_1 be the time for the first event. Sequentially, let T_n denote the time between the $(n-1)$-th and the n-th event. Then $\{T_n\}_{n\geq 1}$ are i.i.d. exponential random variables with parameter λ, e.g.

$$\mathbb{P}(T_n \leq t) = 1 - e^{-\lambda t}.$$

no event

$0 \quad s \quad t+s$

Proof:

$$P(T_1 \leq t) = P(N(t) \geq 1)$$

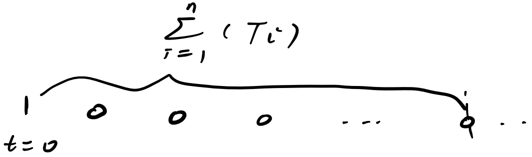
$$P(T_1 > t) = P(N(t) = 0) = \frac{(\lambda t)^0}{0!} e^{-\lambda t} = e^{-\lambda t}, \quad t \geq 0.$$

$$P(T_2 > t \mid T_1 = s) = P(N(t+s) - N(s) = 0 \mid N(s) = 1)$$

$$= P(N(t+s) - N(s) = 0)$$

$$= P(N(t) = 0) = e^{-\lambda t}.$$

Poisson process



Arrival time for Poisson process:

Poisson-Gamma relationship

Consider a Poisson process $\{N(t)\}_{t \geq 0}$ with intensity λ, then the total time until n events is $\sum_{i=1}^{n} T_i \sim \Gamma(n, \lambda)$.

Proof:

T_i iid $\sim$ $Exp(\lambda)$.

$\hookrightarrow \Gamma(1, \lambda)$.

$$\sum_{i=1}^{n} T_i \sim \Gamma(n, \lambda)$$

Poisson process

$$= \frac{P(\text{Poisson}(\lambda(s-t)) = 0)\, P(\text{Poisson}(\lambda t) = 1)}{P(\text{Poisson}(\lambda s) = 1)}$$

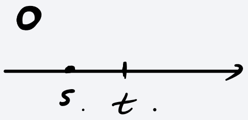
$$= \frac{e^{-\lambda(s-t)}\, \lambda t\, e^{-\lambda t}}{\lambda s\, e^{-\lambda s}} = \frac{t}{s}.$$

Useful Properties:

$T_1 \mid N(s) = 1 \sim U[0, s]$

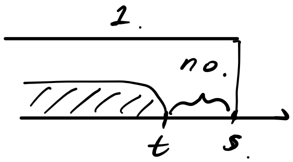
Consider a Poisson process $\{N(t)\}_{t \geq 0}$ with intensity λ, then

$$\underline{\mathbb{P}(T_1 < t \mid N(s) = 1)} = \frac{t}{s}, \quad t < s.$$



Proof:

$s \leq t$ $\quad P(T_1 < t \mid N(s) = 1) = 1$



$t < s.$ $\quad P(T_1 < t \mid N(s) = 1)$

$$= P(N(t) = 1,\ N(s) - N(t) = 0 \mid N(s) = 1) \qquad N(t) = 1$$

$$= P(N(s) - N(t) = 0 \mid N(s) = 1) \quad = \frac{P(N(s) - N(t) = 0)}{P(N(s) = 1)}$$

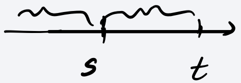
$$= \frac{P(N(s) - N(t) = 0,\ N(s) = 1)}{P(N(s) = 1)}$$

UNIVERSITY OF TORONTO

Poisson process

$N(s) \mid N(t) = n \sim B(n, p = \frac{s}{t})$ for $s < t$

Consider a Poisson process $\{N(t)\}_{t\geq 0}$ with intensity λ, then for $s < t$, $N(s)$ $N(t) - N(s)$



$$\underline{N(s)} \mid \underline{N(t)} = n \sim B(n, p = \frac{s}{t}).$$

Proof: $P(N(s) = k \mid N(t) = n) \longrightarrow$ pmf at k.

$$= \frac{P(N(s) = k,\ N(t) = n)}{P(N(t) = n)} = \frac{P(N(s) = k,\ N(t) - N(s) = n-k)}{P(N(t) = n)}$$

$$= \frac{P(N(s) = k)\, P(N(t) - N(s) = n-k)}{P(N(t) = n)} = \frac{\frac{(\lambda s)^k}{k!} e^{-\lambda s} \frac{(\lambda(t-s))^{n-k}}{(n-k)!} e^{-\lambda(t-s)}}{\frac{(\lambda t)^n}{n!} e^{-\lambda t}}$$

$$= \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k} \nearrow B(n, \frac{s}{t})$$

Poisson process

Superposition

If $\{N_1(t)\}_{t\geq 0}$ and $\{N_2(t)\}_{t\geq 0}$ are independent Poisson processes with intensities λ_1 and λ_2, respectively, then $\{N(t) := N_1(t) + N_2(t)\}_{t\geq 0}$ is also a Poisson process with intensity $\lambda_1 + \lambda_2$.

Proof:

- $N(0) = 0$ ✓
- Ind increments. $N(t+s) - N(t)$
  $= N_1(t+s) - N_1(t) + N_2(t+s) - N_2(t)$
- $N(t) \sim \text{Poisson}(\lambda t)$ $\lambda = \lambda_1 + \lambda_2$. By MGF.

Poisson process

Thinning

Let $\{N(t)\}_{t\geq 0}$ be a Poisson process with intensity λ. Suppose each event is independently of type i with probability p_i for $i = 1, \cdots, k$ with $\sum_{i=1}^{k} p_i = 1$. If $N_i(t)$ is the number of events of type i happen up to time t, then $\{N_i(t)\}$ is a Poisson process with rate λp_i.

Poisson process

Thinning

Let $\{N(t)\}_{t\geq 0}$ be a Poisson process with intensity λ. Suppose each event is independently of type i with probability p_i for $i = 1, \cdots, k$ with $\sum_{i=1}^{k} p_i = 1$. If $N_i(t)$ is the number of events of type i happen up to time t, then $\{N_i(t)\}$ is a Poisson process with rate λp_i.

Properties of Poisson process:

Let $\{N(t)\}_{t\geq 0}$ be a Poisson process with intensity λ, then

- $T_1 \mid N(s) = 1 \sim U[0, s]$;
- $N(s) \mid N(t) = n \sim B(n, p = \frac{s}{t})$ for $s < t$;
- Superposition:
- Thinning.

Brownian motion

Brownian motion: an example of process with continuous time and continuous state

Brownian motion

Standard Brownian motion is a continuous-time process $\{B(t)\}_{t\geq 0}$ satisfying that

- $B(0) = 0$;
- Independent increments: for $0 \leq q < \underline{r \leq s} < t$, $\underbrace{B(t) - B(s)}$ and $\underbrace{B(r) - B(q)}$ are independent random variables;
- $B(t+s) - B(s) \sim \mathcal{N}(0, t)$, $s \geq 0, t > 0$;
- $B(t)$ is almost surely continuous.

$[q, r]$ $[s, t]$

$B(t) - B(s)$ ind $\underbrace{B(s) - B(0)}_{B(s)}$

$t \longrightarrow B(t)$

Remark:

Easy to verify the Markov property.

Brownian motion

Useful properties of Brownian motion:

Joint distribution regarding Brownian motion

For $0 < t_1 < \cdots < t_n$, $(B(t_1), B(t_2), \cdots, B(t_n))^\top$ follows a multivariate normal distribution.

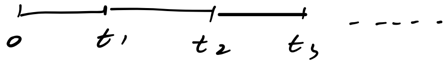
$\forall$ linear combination $(c_1 \cdots c_n)$. $\sum_{i=1}^{n} c_i B(t_i) \sim N(\cdots)$

Proof:

$$\sum_{i=1}^{n} c_i B(t_i) = b_1 B(t_1) + \sum_{i=1}^{n-1} b_{i+1} [B(t_{i+1}) - B(t_i)]$$

$$b_1 = c_1 + c_2 \cdots c_n$$

$$b_{i+1} = \sum_{i+1}^{n} c_i.$$



$N(0, \quad)$

Brownian motion

$Cov(B(s), B(t)) = \min(t, s)$

For a standard Brownian motion $\{B(t)_{t\geq 0}\}$, the covariance satisfies

$$Cov(B(s), B(t)) = \min(t, s).$$

Proof:

$$Cov(B(s), B(t))$$

assume $t < s$.

$$= E(B(s)B(t)) - [\underbrace{E(B(s))}_{0}\underbrace{E(B(t))}_{0}]$$

$$\underbrace{B(s) - B(t)}$$

$$= E((B(s) - B(t) + B(t))B(t))$$

$$= E[(B(s) - B(t))B(t)] + E[B^2(t)]$$

$Var(B(t))$

$$= E[B(s) - B(t)] \cdot E(B(t)) + t$$

$$= t.$$

UNIVERSITY OF TORONTO

Brownian motion

$Cov(B(s), B(t)) = \min(t, s)$

For a standard Brownian motion $\{B(t)_{t \geq 0}\}$, the covariance satisfies

$$Cov(B(s), B(t)) = \min(t, s).$$

Proof:

Remark:

Useful technique: rearrange into independent parts

Brownian motion

Note when

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{MVN}\left(\underline{\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}}, \underline{\begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}}\right),$$

the conditional distribution satisfies

$$X \mid Y = y \sim \mathcal{N}\left(\underline{\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y - \mu_2)}, \underline{(1-\rho^2)\,\sigma_1^2}\right).$$

Brownian motion

Note when

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{MVN}\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}\right),$$

the conditional distribution satisfies

$$X \mid Y = y \sim \mathcal{N}\left(\mu_1 + \underline{\rho\frac{\sigma_1}{\sigma_2}(y - \mu_2)}, (1 - \rho^2)\,\sigma_1^2\right).$$

Conditional distribution regarding Brownian motion

For $0 < s < t$, we have

- $B(s) \mid B(t) = a \sim \mathcal{N}(\frac{s}{t}a, (1 - \frac{s}{t})s)$;
- $B(t) \mid B(s) = a \sim \mathcal{N}(a, t - s)$.

Brownian motion

$$B(s) = a. \qquad B(t)$$

$$0 \quad s \quad t$$

Proof:

$$\begin{pmatrix} B(s) \\ B(t) \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} s & s \\ s & t \end{pmatrix} \right)$$

$$B(s) \mid B(t) = a \sim N\left(\frac{s}{t} a, \left(1 - \frac{s}{t}\right) s \right).$$

$$B(t) \mid B(s) = a \sim N(a, t-s).$$

$$\downarrow.$$

$$\underbrace{B(t) - B(s) \mid B(s) = a}_{B(s) = a.} = B(t) - B(s) \sim N(0, t-s).$$

$$B(t) \mid B(s) = a = \underbrace{B(s) \mid B(s) = a}_{a} + \underbrace{B(t) - B(s) \mid B(s) = a}_{N(0, t-s)}$$

$$\sim N(a, t-s).$$

Brownian motion

$N(\mu, \sigma^2)$ X $N(0,1)$ Z. $\frac{X-\mu}{\sigma} \sim N(0,1)$

Brownian motion with drift

For $\mu \in \mathbb{R}$ and $\sigma > 0$, the process defined by $\{D(t) = \mu t + \sigma B(t)\}$ is called the Brownian motion with drift. μ is the drift parameter and σ^2 is the variance parameter. $X = \sigma Z + \mu$.

Remark:

- $D(0) = 0$;
- $D(t) \sim \mathcal{N}(\mu t, \sigma^2 t^2)$.

UNIVERSITY OF TORONTO

Brownian motion

Brownian motion with drift

For $\mu \in \mathbb{R}$ and $\sigma > 0$, the process defined by $\{D(t) = \mu t + \sigma B(t)\}$ is called the Brownian motion with drift. μ is the drift parameter and σ^2 is the variance parameter.

Remark:

- $D(0) = 0$;
- $D(t) \sim \mathcal{N}(\mu t, \sigma^2 t^2)$.

Example:

Find the probability that Brownian motion with drift takes value between 1 and 2 at time $t = 4$, when $\mu = 0.6, \sigma^2 = 0.25$.

$P(1 \le D(4) \le 2)$

$= P(1 \le 2.4 + 0.5B(4) \le 2)$

$= P(-1.4 \le 0.5B(4) \le -0.4)$.

$\Phi(-0.4) - \Phi(-1.4)$

UNIVERSITY OF TORONTO

Brownian motion

Geometric Brownian Motion

Let $\{D(t) = \mu t + \sigma B(t)\}$ be a Brownian motion with drift, the process $\{G(t) = G(0)e^{D(t)}\}_{t \geq 0}$ is called Geometric Brownian motion, provided that $G(0) > 0$.

Remark:

$\mathbb{E}(G(t)) = G(0)e^{t(\mu + \frac{\sigma^2}{2})}$.

Problem Set

Problem 1: The Poisson process with intensity λ is an example of CTMC.

- Find $P^{(t)}$;
- Compute the generator matrix G.

Problem 2: If $\{N(t)\}_{t \geq 0}$ is a Poisson process with $\lambda = 3$, compute the probability $\mathbb{P}(N(2) = 4, N(4) = 8)$.

Problem 3: Suppose that undergraduate students and graduate students arrive for office hours according to a Poisson process with rate $\lambda_1 = 5$ and $\lambda_2 = 3$ respectively. What is the expected time until the first student arrives?

Problem Set

Problem 4: Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion. Show that the followings are Brownian motions.

- $\{Y(t) = B(t+\alpha) - B(\alpha)\}_{t\geq 0}$ for all $\alpha \geq 0$;
- $\{Y(t) = \alpha B(t/\alpha^2)\}_{t\geq 0}$ for all $\alpha \geq 0$.