



UNIVERSITY OF  
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Statistical Sciences

# DoSS Summer Bootcamp Probability Module 1

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# Roadmap

**A bridge connecting undergraduate probability and graduate probability**

## **Undergraduate-level probability**

- Concrete;
- Examples and scenarios;
- Rely on computation...

# Roadmap

## A bridge connecting undergraduate probability and graduate probability

### Undergraduate-level probability

- Concrete;
- Examples and scenarios;
- Rely on computation...

### Graduate-level probability

- Abstract (measure theory);
- Laws and properties;
- Rely on construction and inference...

# Roadmap

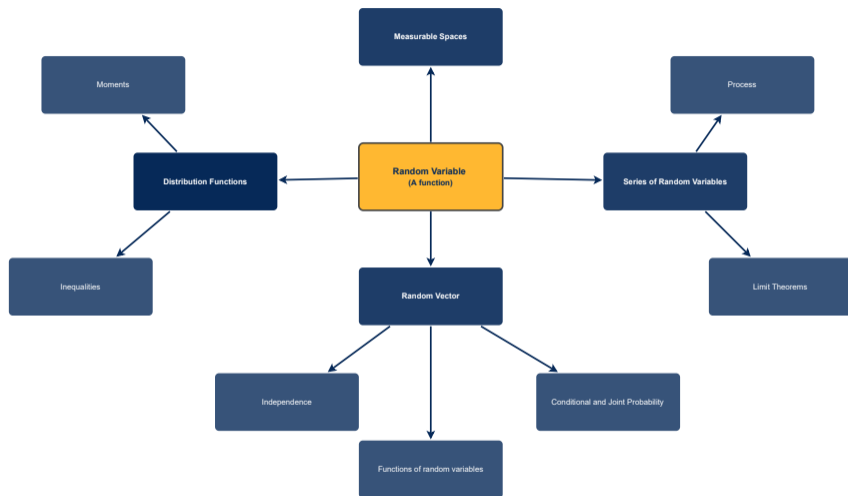


Figure: Roadmap

# Outline

- Measurable spaces
  - ▷ Sample Space
  - ▷  $\sigma$ -algebra
- Probability measures
  - ▷ Measures on  $\sigma$ -field
  - ▷ Basic results
- Conditional probability
  - ▷ Bayes' rule
  - ▷ Law of total probability

# Measurable spaces

## Sample Space

The sample space  $\Omega$  is the set of all possible outcomes of an experiment.

### Examples:

- Toss a coin:  $\{H, T\}$
- Roll a die:  $\{1, 2, 3, 4, 5, 6\}$

# Measurable spaces

## Sample Space

The sample space  $\Omega$  is the set of all possible outcomes of an experiment.

### Examples:

- Toss a coin:  $\{H, T\}$
- Roll a die:  $\{1, 2, 3, 4, 5, 6\}$

## Event

An event is a collection of possible outcomes (subset of the sample space).

### Examples:

- Get head when tossing a coin:  $\{H\}$
- Get an even number when rolling a die:  $\{2, 4, 6\}$

# Measurable spaces

## $\sigma$ -algebra

A  $\sigma$ -algebra ( $\sigma$ -field)  $\mathcal{F}$  on  $\Omega$  is a non-empty collection of subsets of  $\Omega$  such that

- If  $A \in \mathcal{F}$ , then  $A^c \in \mathcal{F}$ ,
- If  $A_1, A_2, \dots \in \mathcal{F}$ , then  $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ .

**Remark:**  $\emptyset, \Omega \in \mathcal{F}$



# Probability measures

## Measures on $\sigma$ -field

A function  $\mu : \mathcal{F} \rightarrow \mathbb{R}^+ \cup \{+\infty\}$  is called a measure if

- $\mu(\emptyset) = 0$ ,
- If  $A_1, A_2, \dots \in \mathcal{F}$  and  $A_i \cap A_j = \emptyset$ , then  $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ .

If  $\mu(\Omega) = 1$ , then  $\mu$  is called a probability measure.

# Probability measures

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If  $\mu(\Omega) = 1$ , then  $\mu$  is called a probability measure.

## Properties:

- Monotonicity:  $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$
- Subadditivity:  $A \subseteq \cup_{i=1}^{\infty} A_i \Rightarrow \mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$
- Continuity from below:  $A_i \nearrow A \Rightarrow \mu(A_i) \nearrow \mu(A)$
- Continuity from above:  $A_i \searrow A$  and  $\mu(A_i) < \infty \Rightarrow \mu(A_i) \searrow \mu(A)$

# Probability measures

Proof of continuity from below:

# Probability measures

Proof of continuity from above:

**Remark:**  $\mu(A_i) < \infty$  is vital.

# Probability measures

## Examples:

$$\Omega = \{\omega_1, \omega_2, \dots\}, A = \{\omega_{a_1}, \dots, \omega_{a_i}, \dots\} \Rightarrow \mu(A) = \sum_{j=1}^{\infty} \mu(\omega_{a_j}).$$

Therefore, we only need to define  $\mu(\omega_j) = p_j \geq 0$ .

If further  $\sum_{i=1}^{\infty} p_j = 1$ , then  $\mu$  is a probability measure.

- Toss a coin:
  
  
  
  
  
  
  
  
  
  
- Roll a die:

# Conditional probability

## Original problem:

- What is the probability of some event  $A$ ?
- $P(A)$  is determined by our probability measure.

## New problem:

- Given that  $B$  happens, what is the probability of some event  $A$ ?
- $P(A | B)$  is the conditional probability of the event  $A$  given  $B$ .

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## Example:

- Roll a die:  $P(\{2\} | \text{even number})$

# Conditional probability

## Bayes' rule

$$P(A | B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

**Remark:** Does conditional probability  $P(\cdot | B)$  satisfy the axioms of a probability measure?



# Conditional probability

## Multiplication rule

$$P(A \cap B) = P(A | B)P(B) = P(B | A)P(A)$$

## Generalization:

## Law of total probability

Let  $A_1, A_2, \dots, A_n$  be a partition of  $\omega$ , such that  $P(A_i) > 0$ , then

$$P(B) = \sum_{i=1}^n P(A_i)P(B | A_i)$$

# Problem Set

**Problem 1:** Prove that for a  $\sigma$ -field  $\mathcal{F}$ , if  $A_1, A_2, \dots \in \mathcal{F}$ , then  $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$ .

**Problem 2:** Prove monotonicity and subadditivity of measure  $\mu$  on  $\sigma$ -field.

**Problem 3:** (Monty Hall problem) Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

(Assumptions: the host will not open the door we picked and the host will only open the door which has a goat.)