UNIVERSITY OF TORONTO | Statistical Sciences

DoSS Summer Bootcamp Probability Module 10

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Recap

Learnt in last module:

- Convergence of functions of random variables
  - Slutsky's theorem
  - Continuous mapping theorem
- Laws of large numbers
  - WLLN
  - SLLN
  - Glivenko-Cantelli theorem
- Central limit theorem

Outline

- Limit Theorems and Counterexamples
  - ▷ Law of Large Numbers
  - ▷ Monotone Convergence Theorem
  - ▷ Dominated Convergence Theorem
  - ▷ More about CLT

Limit Theorems and Counterexamples

Recall: For the law of large numbers to hold, the assumption $E|X| < \infty$ is crucial.

Law of Large Numbers fail for infinite mean i.i.d. random variables

If $X_1 X_2, \ldots$ are i.i.d. to X with $E|X_i| = \infty$, then for $S_n = X_1 + \cdots + X_n$, $P(\lim_{n\to\infty} S_n/n \in (-\infty, \infty)) = 0$.

Proof: Omitted

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Limit Theorems and Counterexamples

Monotone Convergence Theorem

If $X_n \geq c$ and $X_n \nearrow X$, then $EX_n \nearrow EX$

Usage:

Limit Theorems and Counterexamples

Monotone Convergence Theorem

If $X_n \geq 0$ and $X_n \nearrow X$, then $EX_n \nearrow EX$

Counterexample when X_n is not lower bounded:

Limit Theorems and Counterexamples

Dominated Convergence Theorem

If $X_n \to X$ a.s. and $|X_n| \leq Y$ a.s. for all n and Y is integrable, then $EX_n \to EX$

Usage:

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Limit Theorems and Counterexamples

Dominated Convergence Theorem

If $X_n \to X$ a.s. and $|X_n| \leq Y$ a.s. for all n and Y is integrable, then $EX_n \to EX$

Counterexample when X_n is not dominated by an integrable random variable:

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Limit Theorems and Counterexamples

More about CLT: Delta method

Suppose X_n are i.i.d. random variables with $EX_n = 0$, $VAR(X_n) = \sigma^2 > 0$. Let g be a measurable function that is differentiable at 0 with $g'(0) \neq 0$. Then

$$\sqrt{n}\left(g\left(\frac{\sum_{k=1}^{n} X_k}{n} - g(0)\right)\right) \to N(0, \sigma^2 g'(0)^2) \quad \text{weakly.}$$

Proof under stronger assumption: Here, we suppose g is continuously differentiable on $\mathbb{R}$. If you are interested in a general proof refer to Robert Keener's *Theoretical Statistics*.