UNIVERSITY OF TORONTO | Statistical Sciences

DoSS Summer Bootcamp Probability Module 10

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July 28, 2023

Recap

Learnt in last module:

- Convergence of functions of random variables
  - ▷ Slutsky's theorem
  - ▷ Continuous mapping theorem
- Laws of large numbers
  - ▷ WLLN
  - ▷ SLLN
  - ▷ Glivenko-Cantelli theorem
- Central limit theorem

Outline

- Limit Theorems and Counterexamples
  - Law of Large Numbers
  - Monotone Convergence Theorem
  - Dominated Convergence Theorem
  - More about CLT

Limit Theorems and Counterexamples

integrable.

Recall: For the law of large numbers to hold, the assumption $E|X| < \infty$ is crucial.

Law of Large Numbers fail for infinite mean i.i.d. random variables

If $X_1 X_2, \ldots$ are i.i.d. to X with $E|X_i| = \infty$, then for $S_n = X_1 + \cdots + X_n$, $P(\lim_{n\to\infty} S_n/n \in (-\infty, \infty)) = 0$.

Proof: Omitted

Probability of limit of S_n/n exists is zero

$\Leftrightarrow$ S_n/n doesn't converge a.s.

Limit Theorems and Counterexamples

Monotone Convergence Theorem

If $X_n \geq c$ and $X_n \nearrow X$, then $EX_n \nearrow EX$

Usage: Let X_n be $\mathbb{P}(X_n = \frac{1}{n^2}) = p = 1 - \mathbb{P}(X_n = 0)$

Note $0 \leq X_n \leq \frac{1}{n^2}$, $EX_n = \frac{p}{n^2}$

Let $S_n = \sum_{i=1}^{n} X_i$. Then S_n is monotone increasing since $X_n \geq 0$..

Also, $S_n \geq 0$.

Furthermore $S_n \leq \sum_{i=1}^{n} \frac{1}{i^2} \leq \frac{\pi^2}{6} = \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

Therefore $S_n \nearrow \underbrace{Z}_{\text{r.v.}} \leq \frac{\pi^2}{6}$

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Do $E\xi_n \to E\xi$?

$\xi_n \geq 0$, $\xi_n \nearrow \xi$ so we can use the monotone convergence theorem.

$\Rightarrow \lim_{n\to\infty} E\xi_n = E\xi$.

$$E\xi_n = \sum_{i=1}^{n} EX_i = \sum_{i=1}^{n} \frac{p}{i^2} = p\sum_{i=1}^{n} \frac{1}{i^2}.$$

$$\therefore \quad E\xi = p \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i^2} = p \cdot \frac{\pi^2}{6}.$$

Limit Theorems and Counterexamples

Monotone Convergence Theorem

If $X_n \geq \cancel{0}\, c$ and $X_n \nearrow X$, then $EX_n \nearrow EX$

Counterexample when X_n is not lower bounded:

Let X_0 be $\mathbb{P}(X_0 = -2^{i}) = 2^{-i}$ for $i = 1, 2, \ldots$ $X_0 = 0$ otherwise.

Then X_0 is not lower bounded and $EX_0 = -\infty$, $X_0 < 0$

Let $X_n = n^{-1} X_0$.

Then X_n is monotone increasing since

$$X_{n+1} - X_n = \frac{X_0}{n+1} - \frac{X_0}{n} = -\frac{X_0}{n(n+1)} > 0$$

Furthermore, $\lim_{n\to\infty} X_n = 0$ since $\lim_{n\to\infty} X_n = \lim_{n\to\infty} \frac{X_0}{n} = 0$

However, $EX_n = n^{-1} EX_0 = -\infty$

Therefore, $\lim_{n\to\infty} EX_n = -\infty \neq 0 = E \lim_{n\to\infty} X_n$.

$E|Y| < \infty$

Dominated Convergence Theorem

If $X_n \to X$ a.s. and $|X_n| \le Y$ a.s. for all n and Y is integrable, then $EX_n \to EX$

Usage:

X_n is dominated by integrable Y.

Let $M(t) = Ee^{Xt}$ moment generating function of X.

Suppose $M(t) < \infty$ for any $t \in [-\varepsilon, \varepsilon]$.

Then $\frac{d}{dt} M(t) \Big|_{t=0} = EX$

(Proof) For $h \in (-\varepsilon/2, \varepsilon/2)$,

$$\frac{M(h)-M(0)}{h} = E\,\frac{e^{hx}-1}{h}$$

Note that $\displaystyle\lim_{h\to 0} \frac{e^{hx}-1}{h} = X$.

Note that $\displaystyle\left|\frac{e^{hx}-1}{h}\right| = \left|\frac{hx \cdot e^{\xi x}}{h}\right|$ by mean-value theorem where ξ is between 0 and h.

$$= |X|\,e^{\xi X}$$

Note that $\underbrace{|a| \le e^{a} + e^{-a}}$

$\downarrow$.

Therefore,

$$|X|\,e^{\xi X} = \frac{2}{\varepsilon}\cdot\frac{\varepsilon}{2}|X|\cdot e^{\xi X}$$

$$\le \frac{2}{\varepsilon}\cdot\left(e^{\frac{\varepsilon}{2}x} + e^{-\frac{\varepsilon}{2}x}\right)\cdot e^{\xi X}$$

$$= \frac{2}{\varepsilon}\left(e^{(\xi+\frac{\varepsilon}{2})x} + e^{(\xi-\frac{\varepsilon}{2})x}\right)$$

Note that $\xi \pm \frac{\varepsilon}{2} \in (-\varepsilon, \varepsilon)$

$$\le C\cdot\underbrace{\frac{2}{\varepsilon}\left(e^{\varepsilon x} + e^{-\varepsilon x}\right)}_{\text{integrable.}}$$

Therefore, we can use the dominated convergence thm to $\dfrac{e^{hx}-1}{h}$.

$$\Rightarrow \underbrace{\lim_{h\to 0} E\,\frac{e^{hx}-1}{h}}_{\substack{\| \\ \lim_{h\to 0}\frac{M(h)-M(0)}{h} \\ \| \\ M'(0)}} = E \lim_{h\to 0}\frac{e^{hx}-1}{h} = EX.$$

Limit Theorems and Counterexamples

Dominated Convergence Theorem

If $X_n \to X$ a.s. and $|X_n| \leq Y$ a.s. for all n and Y is integrable, then $EX_n \to EX$

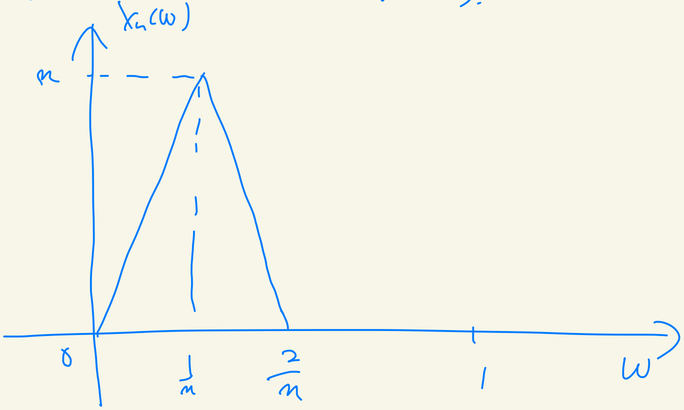
Counterexample when X_n is not dominated by an integrable random variable:

1) the Conterexample for monotone convergence theorem.

2) Let $\Omega = (0,1)$ with $\mathbb{P}(\omega \in (a,b]) = b-a$ if $0<a<b<1$

("uniform measure)

Then define $X_n(\omega)$ as follows.



for all $\omega \in (0,1) = \Omega$, $\lim_{n\to\infty} X_n(\omega) = 0$

while EX_n = the area of triangle $= \frac{1}{2} \cdot n \cdot \frac{2}{n} = 1$

Therefore $\lim_{n\to\infty} EX_n = 1 \neq 0 = E \lim_{n\to\infty} X_n$ //

Limit Theorems and Counterexamples

More about CLT: Delta method

Suppose X_n are i.i.d. random variables with $EX_n = 0$, $VAR(X_n) = \sigma^2 > 0$. Let g be a measurable function that is differentiable at 0 with $g'(0) \neq 0$. Then

$$\sqrt{n}\left(g\left(\frac{\sum_{k=1}^{n} X_k}{n}\right) - g(0)\right) \to N(0, \sigma^2 g'(0)^2) \quad \text{weakly.}$$

Proof under stronger assumption: Here, we suppose g is continuously differentiable on $\mathbb{R}$. If you are interested in a general proof refer to Robert Keener's *Theoretical Statistics*.

$$\sqrt{n}\left(g(\bar{x}) - g(0)\right) \xrightarrow{d} N\left(0, \sigma^2 g'(0)^2\right)$$

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By the mean-value theorem, there exists c_n s.t.

$$g(\bar{x}) - g(0) = g'(c_n)\bar{X}$$ where c_n is between 0 and $\bar{X}$.

By the strong law of large numbers, $\lim_{n\to\infty} \bar{X} = 0$ a.s.

Since c_n is between 0 and $\bar{X}$, $\lim_{n\to\infty} c_n = 0$ a.s.

By the continuity of g', $\lim_{n\to\infty} g'(c_n) = g'(0)$ almost surely.

$$\sqrt{n}\left(g(\bar{x}) - g(0)\right) = \underbrace{g'(c_n)}_{\substack{\to g(0) \\ \text{a.s.}}} \cdot \underbrace{\sqrt{n}\,\bar{X}}_{\substack{\to N(0,\sigma^2) \\ \text{by CLT}}}$$

by Slutsky's theorem.

$$\xrightarrow{d} N\left(0, \sigma^2 g'(0)^2\right)$$