UNIVERSITY OF TORONTO | Statistical Sciences

DoSS Summer Bootcamp Probability Module 6

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Recap

Learnt in last module:

- Moments
  - ▷ Expectation, Raw moments, central moments
  - ▷ Moment-generating functions
- Change-of-variables using MGF
  - ▷ Gamma distribution
  - ▷ Chi square distribution
- Conditional expectation
  - ▷ Law of total expectation
  - ▷ Law of total variance

Outline

- Covariance
  - Covariance as an inner product
  - Correlation
  - Cauchy-Schwarz inequality
  - Uncorrelatedness and Independence
- Concentration
  - Markov's inequality
  - Chebyshev's inequality
  - Chernoff bounds

Covariance

Recall the property of expectation:

$$\mathbb{E}(X+Y)=\mathbb{E}(X)+\mathbb{E}(Y).$$

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Covariance

Recall the property of expectation:

$$\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y).$$

What about the variance?

$$\begin{aligned} Var(X+Y) &= \mathbb{E}(X+Y-\mathbb{E}(X)-\mathbb{E}(Y))^2 \\ &= \mathbb{E}(X-\mathbb{E}(X))^2 + \mathbb{E}(Y-\mathbb{E}(Y))^2 + 2\mathbb{E}((X-\mathbb{E}(X))(Y-\mathbb{E}(Y))) \\ &= Var(X) + Var(Y) + 2\,\underbrace{\mathbb{E}((X-\mathbb{E}(X))(Y-\mathbb{E}(Y)))}_{?} \end{aligned}$$

Covariance

Intuition:

A measure of how much X, Y change together.

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A measure of how much X, Y change together.

Covariance

For two jointly distributed real-valued random variables X, Y with finite second moments, the covariance is defined as

$$Cov(X,Y) = \mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))).$$

Simplification:

$$Cov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y).$$

$$Cov(X,Y) = EXY - EXEY - EYEX + EXEY$$

$$= EXY - EX \cdot EY.$$

Covariance

Properties:

- $Cov(X, X) = Var(X) \geq 0$;
- $Cov(X, a) = 0$, $\quad a$ is a constant;
- $Cov(X, Y) = Cov(Y, X)$;
- $Cov(X + a, Y + b) = Cov(X, Y)$;
- $Cov(aX, bY) = abCov(X, Y)$.

$$Cov(X, a) = E\left(X - EX\right) \cdot \big(\underbrace{a - Ea}_{=0}\big) = 0$$

$$Cov(X+a, Y+b)$$
$$= E\left(X+a - E(X+a)\right) \cdot \left(Y+b - E(Y+b)\right)$$
$$= E\left(X - EX\right) \cdot \left(Y - EY\right) = Cov(X, Y)$$

$$Cov(aX, bY) = E\left(aX - EaX\right)\left(bY - EbY\right)$$
$$= ab\, E(X - EX)(Y - EY) = ab\, Cov(X, Y)$$

Covariance

Properties:

- $Cov(X, X) = Var(X) \geq 0;$ (I)
- $Cov(X, a) = 0,$ a is a constant;
- $Cov(X, Y) = Cov(Y, X);$
- $Cov(X + a, Y + b) = Cov(X, Y);$ (IV)
- $Cov(aX, bY) = abCov(X, Y).$

Corollary about variance:

$$Var(aX + b) = a^2 Var(X).$$

$$Var(ax+b) \overset{(I)}{=} Cov(ax+b, ax+b) \overset{(IV)}{=} Cov(ax, ax) \overset{(V)}{=} a^2 Cov(x,x) \overset{(I)}{=} a^2 Var(x)$$

Covariance

Relate covariance to inner product:

Inner product (not rigorous)

Inner product is a operator from a vector space V to a field F (use $\mathbb{R}$ here as an example): $< \cdot, \cdot >$: $V \times V \to \mathbb{R}$ that satisfies:

- Symmetry: $< x, y > = < y, x >$;
- Linearity in the first argument: $< ax + by, z > = a < x, z > + b < y, z >$;
- Positive-definiteness: $< x, x > \geq 0$, and $< x, x > = 0 \Leftrightarrow x = 0$

$V = \underbrace{L^2 \text{ space.}}_{\text{a space of}}$ square-integrable random variables.

Covariance

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Remark:

Covariance defines an inner product over the quotient vector space obtained by taking the subspace of random variables with finite second moment and identifying any two that differ by a constant.

Covariance

Properties inherited from the inner product space

Recall in Euclidean vector space:

- $< x, y >= x^\top y = \sum_{i=1}^{n} x_i y_i$;
- $||x||_2 = \sqrt{< x, x >}$;
- $< x, y >= ||x||_2 \cdot ||y||_2 \cos(\theta)$.

$\cos\theta = \dfrac{\langle x, y \rangle}{\|x\|_2 \, \|y\|_2}$.

Respectively:

- $< X, Y >= Cov(X, Y)$;
- $||X|| = \sqrt{Var(X)}$;

Covariance

A substitute for $\cos(\theta)$**:**

Correlation

For two jointly distributed real-valued random variables X, Y with finite second moments, the correlation is defined as

$$Corr(X, Y) = \rho_{XY} = \frac{Cov(X, Y)}{\sqrt{Var(X) \cdot Var(Y)}}.$$

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Covariance

A substitute for $\cos(\theta)$**:**

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Uncorrelatedness:

$$X, Y \text{ uncorrelated} \quad \Leftrightarrow \quad Corr(X, Y) = 0.$$

Covariance $|ax+by| = |\langle \binom{a}{b}, \binom{x}{y} \rangle| \le \sqrt{a^2+b^2}\sqrt{x^2+y^2}$

Cauchy-Schwarz inequality

$$|Cov(X,Y)| \le \sqrt{Var(X)Var(Y)}.$$

Proof: Let $X - EX = \hat{X}$, $Y - E = \tilde{Y}$.

$$0 \le E(\hat{X} + t\tilde{Y})^2 = E\hat{X}^2 + 2t\,E\hat{X}\cdot E\tilde{Y} + t^2 E\tilde{Y}^2.$$

$$= Var(X) + 2\,Cov(X,Y)\cdot t + Var(Y)\,t^2.$$

This holds for any $t \in \mathbb{R}$,

$$D/4 = Cov(X,Y)^2 - Var(X)Var(Y) \le 0$$

$$\therefore \quad Cov(X,Y)^2 \le Var(X)Var(Y)$$

Covariance

Uncorrelatedness and Independence:

Observe the relationship:

$$Corr(X, Y) = 0 \quad \Leftrightarrow \quad Cov(X, Y) = 0 \quad \Leftrightarrow \quad \mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(X)$$

This can happen
when X and Y are independent

Covariance

Uncorrelatedness and Independence:

Observe the relationship:

$$Corr(X, Y) = 0 \quad \Leftrightarrow \quad Cov(X, Y) = 0 \quad \Leftrightarrow \quad \mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(X)$$

Conclusions:

- Independence $\Rightarrow$ Uncorrelatedness
- Uncorrelatedness $\not\Longrightarrow$ Independence

Remark:
Independence is a very strong assumption/property on the distribution.

Covariance

Special case: multivariate normal

Multivariate normal

A k-dimensional random vector $\mathbf{X} = (X_1, X_2, \cdots, X_k)^\top$ follows a multivariate normal distribution $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, if

mean   covariance. $\Sigma \in \mathbb{R}^{k \times k}$

$$f_{\mathbf{X}}(x_1, \ldots, x_k) = \frac{\exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})\right)}{\sqrt{(2\pi)^k |\boldsymbol{\Sigma}|}},$$

where $\boldsymbol{\mu} = \mathbb{E}[\mathbf{X}] = (\mathbb{E}[X_1], \mathbb{E}[X_2], \ldots, \mathbb{E}[X_k])^\top$, and $[\boldsymbol{\Sigma}]_{i,j} = \Sigma_{i,j} = Cov(X_i, X_j)$.

$\iff \quad \Sigma = \mathbb{E}\,(x-\mu)(x-\mu)^T$

Observation:
The distribution is decided by the covariance structure.

$$f_X(x_1,\cdots,x_d) = (2\pi)^{-\frac{d}{2}} \, |\det\Sigma|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)$$

Note Σ is symmetric matrix.

$\Rightarrow$ There exists an $\underbrace{\text{orthogonal matrix}}_{(VV^T = V^TV = I)}$ V and diagonal $\Lambda = \begin{pmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_d^2 \end{pmatrix}$

when $\lambda_i \geq 0$ (We can say this because Σ is positive semi definite)

Further assume that $\lambda_i > 0$ (it ensures Σ and Λ to be invertible)

s.t. $\Sigma V = V\Lambda \Leftrightarrow \underbrace{\Sigma = V\Lambda V^T}_{\text{spectral decomposition of symmetric matrix.}}$

$$\Leftrightarrow \Sigma^{-1} = V\Lambda^{-1}V^T$$

$$(x-\mu)^T\Sigma^{-1}(x-\mu) = (x-\mu)^T V \Lambda^{-1} \underbrace{V^T(x-\mu)}_{=z}$$

change variables by $z = V^T(x-\mu)$.

Note that Jacobian of $x \to z$ is 1

since $|\det V| = 1$.

therefore

$$p(z) = p(x) = (2\pi)^{-\frac{k}{2}} \underbrace{|\det \Lambda|}^{-\frac{1}{2}} \exp\left[-\frac{1}{2} z^T \Lambda^{-1} z\right]$$

$|\det \Lambda| = |\det \Sigma|$

due to $|\det V| = 1$

Recall that $\Lambda = \begin{pmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_k^2 \end{pmatrix}$,

So, $$p(z) = (2\pi)^{-\frac{k}{2}} \left| \prod_{i=1}^{k} \lambda_i^2 \right|^{-\frac{1}{2}} \exp\left[- \sum_{i=1}^{k} \frac{z_i^2}{2\lambda_i^2}\right]$$

$$= \prod_{i=1}^{k} \left[\underbrace{\frac{1}{\sqrt{2\pi}\,\lambda_i} \exp\left(-\frac{z_i^2}{2\lambda_i^2}\right)}_{\text{1-dimensional normal}} \right]$$

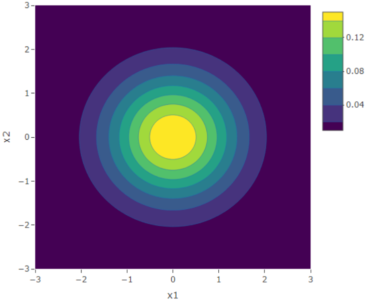
Covariance

$$X_i, i = 1, \cdots k \text{ independent} \Leftrightarrow f_{\mathbf{X}}(x_1, \ldots, x_k) = \prod_{i=1}^{k} f_{X_i}(x_i)$$

$$\Leftrightarrow \underline{\boldsymbol{\Sigma} = I_k} \Leftrightarrow Cov(X_i, X_j) = 0, i \neq j.$$

Example:

- $Corr(X, Y) = 0$



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Covariance

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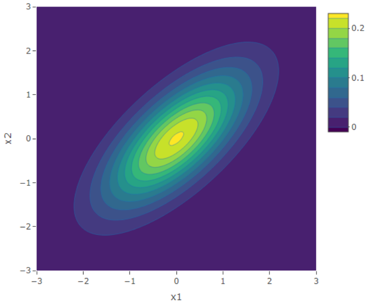
$$\Leftrightarrow \mathbf{\Sigma} = I_k \Leftrightarrow Cov(X_i, X_j) = 0, i \neq j.$$

Example:

- $Corr(X, Y) = 0.7$



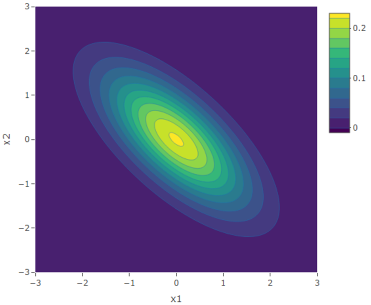
Covariance

$$X_i, i = 1, \cdots k \text{ independent} \Leftrightarrow f_{\mathbf{X}}(x_1, \dots, x_k) = \prod_{i=1}^{k} f_{X_i}(x_i)$$

$$\Leftrightarrow \mathbf{\Sigma} = I_k \Leftrightarrow Cov(X_i, X_j) = 0, i \neq j.$$

Example:

- $Corr(X, Y) = -0.7$



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Concentration

Measures of a distribution:

- $\mathbb{E}(X^k)$, $\mathbb{E}(X)$, $Var(X)$;
- $Cov(X, Y)$ and $Corr(X, Y)$.

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- $\mathbb{E}(X^k)$, $\mathbb{E}(X)$, $Var(X)$;
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Tail probability: $\mathbf{P}(|X| > t)$

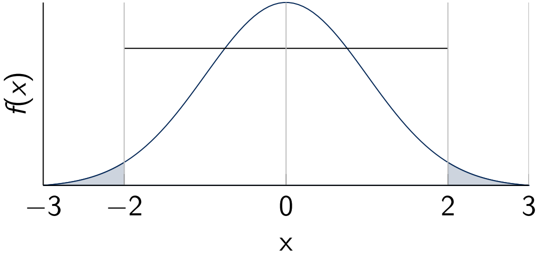

Figure: Probability density function of $\mathcal{N}(0, 1)$

Concentration

Concentration inequalities:

- Markov inequality
- Chebyshev inequality
- Chernoff bounds

Concentration

Concentration inequalities:

- Markov inequality
- Chebyshev inequality
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Markov inequality

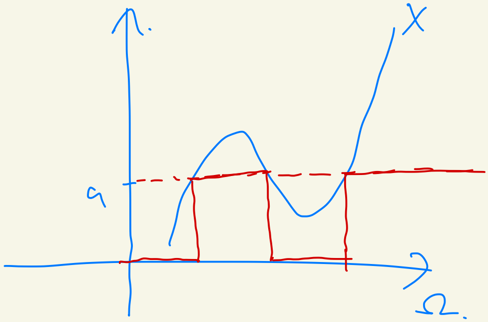
Let X be a random variable that is non-negative (almost surely). Then, for every constant $a > 0$,

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}.$$

Proof: We use monotonicity of expectation, i.e.

if $X \geq Y$, then $EX \geq EY$.

$Y = a \, \mathbb{1}[X \geq a]$



Then $X \geq Y$

So $\mathbb{E} X \geq \mathbb{E} Y$

$= a \, \mathbb{E} \mathbb{1}[X \geq a]$

$= a \, \mathbb{P}(X \geq a)$

Since $a > 0$, $\mathbb{P}(X \geq a) \leq \dfrac{\mathbb{E} X}{a}$.

Concentration

Markov inequality (continued)

Let X be a random variable, then for every constant $a > 0$,

$$\mathbb{P}(|X| \geq a) \leq \frac{\mathbb{E}(|X|)}{a}.$$

A more general conclusion:

Markov inequality (continued)

Let X be a random variable, if $\Phi(x)$ is monotonically increasing on $[0, \infty)$, then for every constant $a > 0$,

$$\mathbb{P}(|X| \geq a) = \mathbb{P}(\Phi(|X|) \geq \Phi(a)) \leq \frac{\mathbb{E}(\Phi(|X|))}{\Phi(a)}.$$

due to Markov inequality

Concentration

Chebyshev inequality

Let X be a random variable with finite expectation $\mathbb{E}(X)$ and variance $Var(X)$, then for every constant $a > 0$,

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq a) \leq \frac{Var(X)}{a^2},$$

or equivalently,

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq a\sqrt{Var(X)}) \leq \frac{1}{a^2}.$$

Example:

Take $a = 2$,

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq 2\sqrt{Var(X)}) \leq \frac{1}{4}.$$

We can show Chebyshev inequality by applying Markov inequality with X replaced by $X - EX$ and $\phi(x) = x^2$.

Concentration

Chernoff bound (general)

Let X be a random variable, then for $t \geq 0$,

Markov inequality

$$\mathbb{P}(X \geq a) = \mathbb{P}(e^{t \cdot X} \geq e^{t \cdot a}) \leq \frac{\mathbb{E}\left[e^{t \cdot X}\right]}{e^{t \cdot a}},$$

due to $\{X \geq a\} = \{t \cdot X \geq t \cdot a\} = \{e^{tX} \geq e^{ta}\}$

and

$$\mathbb{P}(X \geq a) \leq \inf_{t \geq 0} \frac{\mathbb{E}\left[e^{t \cdot X}\right]}{e^{t \cdot a}}.$$

minimum of RHS.

Remark:

This is especially useful when considering $X = \sum_{i=1}^{n} X_i$ with X_i's independent,

$$\mathbb{P}(X \geq a) \leq \inf_{t \geq 0} \frac{\mathbb{E}\left[\prod_i e^{t \cdot X_i}\right]}{e^{t \cdot a}} = \inf_{t \geq 0} e^{-t \cdot a} \prod_i \mathbb{E}\left[e^{t \cdot X_i}\right].$$

In particular, if $X_1, \dots, X_n \overset{\text{i.i.d}}{\sim} Z$

$$\mathbb{P}(X \geq a) \leq \inf_{t>0} \left(\mathbb{E}[e^{tZ}] \right)^n / e^{t \cdot a}.$$

e.g.) $X_i \overset{\text{i.i.d}}{\sim} \text{Bern}(1/2)$

$$\mathbb{E}[e^{t \cdot X_i}] = \frac{e^t + e^{-t}}{2}$$

RHS of Chernoff bound $= \underbrace{e^{-ta} \cdot \left(\frac{e^t + e^{-t}}{2} \right)^n}$.

can find minimum ~~of~~

by differentiating with t.

Problem Set

Problem 1: Let

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases},$$

compute $Cov(X, Y)$.

Problem 2: For $X \sim \mathcal{N}(0, 1)$, compute the Chernoff bound.