UNIVERSITY OF TORONTO | Statistical Sciences

DoSS Summer Bootcamp Probability Module 8

Ichiro Hashimoto

University of Toronto

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Recap

Learnt in last module:

- Stochastic convergence
  - Convergence in distribution $\to$ CDF — CLT
  - Convergence in probability $\to$ $\mathbb{P}(|X_n - X| > \varepsilon) \to 0$ for any $\varepsilon > 0$
  - Convergence almost surely $\to$ $\mathbb{P}(\lim X_n = X) = 1$
  - Convergence in L^p $\to$ moment $E|X_n - X|^p \to 0$
  - Relationship between convergences

$$
\begin{tikzcd}
\underbrace{L^q \to L^p}_{q>p>0} \arrow[dr] & & \\
& \text{Cvg in probability} \arrow[r] & \text{Cvg in distribution.} \\
\text{a.s.} \arrow[ur] & &
\end{tikzcd}
$$

For all of these, their converse fails.

Outline

- Relationship between convergences and counterexamples
  - ▷ Monotonicity of L^p Convergence
  - ▷ L^p convergence implies Convergence in Probability
  - ▷ a.s. Convergence implies Convergence in Probability
  - ▷ Convergence in Probability implies Convergence in distribution

Relationship between convergences and counterexamples

Recall: A random variable $X \in L^p$ if $\|X\|_{L^p} = (E|X|^p)^{1/p} < \infty$.

$X_n \to X$ in L^p if $\lim_{n\to\infty} \|X_n - X\|_{L^p} = 0$

Monotonicity of L^p Convergence

If $q > p > 0$, L^q convergence implies L^p convergence

Proof: Suppose $X_n \to X$ in L^q. Then $E|X_n - X|^q \to 0$ as $n \to \infty$

By Jensen's inequality with $\varphi(x) = |x|^{\frac{q}{p}}$ (convex when $q > p > 0$),

$$E|X_n - X|^q = E(|X_n - X|^p)^{\frac{q}{p}} \geq (E|X_n - X|^p)^{\frac{q}{p}}$$

$$\therefore E|X_n - X|^p \leq (E|X_n - X|^q)^{\frac{p}{q}} \to 0$$

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Relationship between convergences and counterexamples

Recall: A random variable $X \in L^p$ if $\|X\|_{L^p} = (E|X|^p)^{1/p} < \infty$.

$X_n \to X$ in L^p if $\lim_{n\to\infty} \|X_n - X\|_{L^p} = 0$

Monotonicity of L^p Convergence

If $q > p > 0$, L^q convergence implies L^p convergence

Counterexample to the Converse:

Let $\mathbb{P}(X_n = n^a) = \frac{1}{n} = 1 - \mathbb{P}(X_n = 0)$.

Then $E|X_n|^p = n^{ap-1}$, $E|X_n|^q = n^{aq-1}$

Then if we choose a s.t. $ap-1 < 0 < aq-1$, $\frac{1}{q} < a < \frac{1}{p}$

we have $E|X_n|^p \to 0$ while $E|X_n|^q \to \infty$

That means $X_n \to 0$ in L^p but $X_n \to 0$ in L^q fails.

Relationship between convergences and counterexamples

Recall: X_n converges to X in probability if for any $\epsilon > 0$ $\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0$.

L^p convergence implies Convergence in Probability

If $X_n \to X$ in L^p, then $X_n \to X$ in probability.

Proof: By Markov inequality

$$\mathbb{P}(|X_n - X| > \varepsilon) = \mathbb{P}(|X_n - X|^p > \varepsilon^p) \leq \underbrace{\varepsilon^{-p}\, \mathbb{E}|X_n - X|^p \to 0}_{\text{by } X_n \to X \text{ in } L^p}$$

Relationship between convergences and counterexamples

Recall: X_n converges to X in probability if for any $\epsilon > 0$ $\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0$.

L^p convergence implies Convergence in Probability

If $X_n \to X$ in L^p, then $X_n \to X$ in probability.

Counterexample to the Converse:

$\mathbb{P}(X_n = n^{1/p}) = \frac{1}{n} = 1 - \mathbb{P}(X_n = 0)$

$\mathbb{P}(|X_n - 0| > \varepsilon) = \frac{1}{n}$ for sufficiently large n.

Hence $\lim_{n\to\infty} \mathbb{P}(|X_n - 0| > \varepsilon) = 0$ for any $\varepsilon > 0$.

That means $X_n \to 0$ in probability.

However, $E|X_n - 0|^p = (n^{1/p})^p \cdot \frac{1}{n} + 0 = 1 \not\to 0$

So, X_n does not converge in L^p.

Relationship between convergences and counterexamples

Recall: X_n converges to X in probability if for any $\epsilon > 0$ $\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0$.

a.s. Convergence implies Convergence in Probability

If $X_n \to X$ almost surely, then $X_n \to X$ in probability.

Proof: $X_n \to X$ almost surely implies that.

for $\forall \varepsilon > 0$, almost surely, $|X_n - X| > \varepsilon$ for finitely many n.

(same as $|X_n - X| < \varepsilon$ for sufficiently large n).

i.e. $\mathbb{P}($ $|X_n - X| > \varepsilon$ for infinitely many $n) = 0$ for $\forall \varepsilon > 0$.

Note that

$$\limsup_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) \leq \mathbb{P}(|X_n - X| > \varepsilon \text{ for infinitely many } n)$$

$$= 0$$

$$\therefore \quad \lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0 \quad //.$$

<u>Thm</u> For X_n, X, the following are equivalent.

(i) $X_n \to X$ in probability

(ii) For any subsequence of $\{X_n\}$, we can find further subsequence that converges to X a.s.

Relationship between convergences and counterexamples

Recall: X_n converges to X in probability if for any $\epsilon > 0$ $\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0$.

a.s. Convergence implies Convergence in Probability

If $X_n \to X$ almost surely, then $X_n \to X$ in probability.

Counterexample to the Converse:

Let $\Omega = (0,1)$, $\mathcal{F} = \mathcal{B}(\Omega)$ Borel sets, $\mathbb{P}$ on Ω by uniform measure.

i.e. $\mathbb{P}\big((a,b)\big) = b-a$.

if $0<a<b<1$

Define $X_{n,m}(\omega) = \begin{cases} 1 & \text{if } \omega \in \left(\frac{m}{n}, \frac{m+1}{n}\right] \\ 0 & \text{otherwise.} \end{cases}$ $\quad 0 \leq m \leq n-1$

This makes $\mathbb{P}(X_{n,m} = 1) = \frac{1}{n} = 1 - \mathbb{P}(X_{n,m})$.

So, $X_{n,m} \to 0$ in probability as $n \to \infty$.

But for each $\omega \in (0,1)$

$X_{n,m}(\omega) = 1$. for infinitely many n, m.

That means $X_{n,m}$ does not converge to 0

for any point on Ω.

Relationship between convergences and counterexamples

Recall: X_n converges to X in distribution if for any continuity point x of $P(X \leq x)$, $\lim_{n\to\infty} P(X_n \leq x) = P(X \leq x)$ holds.

Convergence in Probability implies Convergence in Distribution

If $X_n \to X$ in probability, then $X_n \to X$ in distribution.

Proof: Omit

Relationship between convergences and counterexamples

Recall: X_n converges to X in distribution if for any continuity point x of $P(X \leq x)$, $\lim_{n\to\infty} P(X_n \leq x) = P(X \leq x)$ holds.

Convergence in Probability implies Convergence in Distribution

If $X_n \to X$ in probability, then $X_n \to X$ in distribution.

Counterexample to the Converse:

Let $X \sim$ symmetric Bernoulli $(\frac{1}{2})$

i.e. $\mathbb{P}(X = \pm 1) = \frac{1}{2}$.

Define $X_n = (-1)^n X$

Then $X_n =_d X$ for any n since $-X =_d X$

That means $X_n \to X$ in distribution.

However, for any odd n,

$$\mathbb{P}(X_n \neq X) = 1 \quad \text{since } X_n = -X \neq X$$

Therefore, X_n does not converge to X in probability.

$$\text{constant}$$

Thm If $X_n \to C$ in distribution, then

$X_n \to C$ in probability.

Cor If the limit is constant, convergence in probability and convergence in distribution are equivalent.

Relationship between convergences and counterexamples

Recall: X_n converges to X in distribution if for any continuity point x of $P(X \leq x)$, $\lim_{n\to\infty} P(X_n \leq x) = P(X \leq x)$ holds.

Convergence in Probability implies Convergence in Distribution

If $X_n \to X$ in probability, then $X_n \to X$ in distribution.

Special case when the Converse holds: