Problem 1

The Robarts library has recently added a new printer which turns out to be defective. The letter "U" has a 30% chance of being printed out as "V", and the letter "V" has a 10% chance of being printed out as "U". Each letter is printed out independently, and all other letters are always correctly printed.
The librarian uses "UNIVERSITY OF TORONTO" as a test phrase, and will make a complaint to the printer factory immediately after the third incorrectly printed test phrase, calculate the probability that there are fewer correctly printed phrases than incorrectly printed phrases when the complaint is made.

Solution:

Proof. Denote A as the event "There are fewer correctly printed phrases than incorrectly printed phrases when the complaint is made", then the event is equivalent to "When the third incorrectly printed test phrase appears, there are at most 2 correctly printed ones".
Write N_C as number of correctly printed phrases when the complaint is made, and N_I as number of incorrectly printed phrases, respectively. Then let

$$p = \mathbb{P}(\text{The test phrase is correctly printed}) = (1 - 0.3)(1 - 0.1) = 0.63,$$

we have

$$\begin{aligned}\mathbb{P}(A) &= \mathbb{P}(N_I = 3, N_C = 0) + \mathbb{P}(N_I = 3, N_C = 1) + \mathbb{P}(N_I = 3, N_C = 2) \\ &= p^0(1-p)^3 + \binom{3}{1} p^1(1-p)^3 + \binom{4}{2} p^2(1-p)^3 \\ &\approx 0.267.\end{aligned}$$

PS: You can also formulate the question by negative binomial distribution. ∎

Problem 2

Compute the mode of Negative Binomial distribution with parameter r and p.
(Hint: consider $\mathbb{P}(X = k+1)/\mathbb{P}(X = k)$)

Solution:

Proof.

$$\begin{aligned}\frac{\mathbb{P}(X = k+1)}{\mathbb{P}(X = k)} &= \frac{\binom{k}{r-1} p^r (1-p)^{k+1-r}}{\binom{k-1}{r-1} p^r (1-p)^{k-r}} \\ &= \frac{k}{k-r+1} \cdot (1-p),\end{aligned}$$

which is monotonically decreasing regarding k.
Let $\frac{\mathbb{P}(X=k+1)}{\mathbb{P}(X=k)} = 1$, then $k = \frac{r-1}{p}$. Therefore, if $\frac{r-1}{p}$ is integer, then the mode is $\frac{r-1}{p}$ or $\frac{r-1}{p}+1$, otherwise, the mode is $\lceil \frac{r-1}{p} \rceil$. ∎

Problem 3

Show that normal distribution belongs to the exponential family.

Solution:

Proof. Consider the parameter $\theta = (\mu, \sigma^2)^\top$, then

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{(x-\mu)^2}{2\sigma^2}) \\ &= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} \exp(-\frac{1}{2\sigma^2}x^2 + \frac{\mu}{\sigma^2}x - \frac{\mu^2}{2\sigma^2}) \\ &= \frac{1}{\sqrt{2\pi}} \cdot \exp(-\frac{1}{2\sigma^2}x^2 + \frac{\mu}{\sigma^2}x - \frac{\mu^2}{2\sigma^2} - \frac{1}{2}\log(\sigma^2)). \end{aligned}$$

Let $h(x) = \frac{1}{\sqrt{2\pi}}$, $\eta(\theta) = (-\frac{1}{2\sigma^2}, \frac{\mu}{\sigma^2})^\top$, $T(x) = (x^2, x)$, $A(\theta) = \frac{\mu^2}{2\sigma^2} + \frac{1}{2}\log(\sigma^2)$, then this satisfies the format of exponential family. ■