UNIVERSITY OF TORONTO | Statistical Sciences

DoSS Summer Bootcamp Probability Module 6

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Recap

Learnt in last module:

- Moments
  - Expectation, Raw moments, central moments
  - Moment-generating functions
- Change-of-variables using MGF
  - Gamma distribution
  - Chi square distribution
- Conditional expectation
  - Law of total expectation
  - Law of total variance

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Outline

- Covariance
  - Covariance as an inner product
  - Correlation
  - Cauchy-Schwarz inequality
  - Uncorrelatedness and Independence
- Concentration
  - Markov's inequality
  - Chebyshev's inequality
  - Chernoff bounds

Covariance

Recall the property of expectation:

$$\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y).$$

Covariance

Recall the property of expectation:

$$\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y).$$

What about the variance?

$$\begin{aligned} Var(X+Y) &= \mathbb{E}(X+Y-\mathbb{E}(X)-\mathbb{E}(Y))^2 \\ &= \mathbb{E}(X-\mathbb{E}(X))^2 + \mathbb{E}(Y-\mathbb{E}(Y))^2 + 2\mathbb{E}((X-\mathbb{E}(X))(Y-\mathbb{E}(Y))) \\ &= Var(X) + Var(Y) + 2\,\underbrace{\mathbb{E}((X-\mathbb{E}(X))(Y-\mathbb{E}(Y)))}_{?} \end{aligned}$$

Covariance

Intuition:

A measure of how much X, Y change together.

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A measure of how much X, Y change together.

Covariance

For two jointly distributed real-valued random variables X, Y with finite second moments, the covariance is defined as

$$Cov(X, Y) = \mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))).$$

$\mathbb{E}X^2 < \infty$

$\mathbb{E}Y^2 < \infty$

Simplification:

$$Cov(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y).$$

$$Cov(X,Y) = \mathbb{E}((X-\mathbb{E}X)(Y-\mathbb{E}Y)) = \mathbb{E}(XY - (\mathbb{E}X)Y - (\mathbb{E}Y)\cdot X + (\mathbb{E}X)(\mathbb{E}Y))$$

$$= \mathbb{E}(XY) - (\mathbb{E}X)\cdot(\mathbb{E}Y) - (\mathbb{E}X)\cdot(\mathbb{E}Y) + (\mathbb{E}X)\cdot(\mathbb{E}Y)$$

Covariance

Properties:

- $Cov(X, X) = Var(X) \geq 0$;
- $Cov(X, a) = 0$, $\quad a$ is a constant;
- $Cov(X, Y) = Cov(Y, X)$;
- $Cov(X + a, Y + b) = Cov(X, Y)$;
- $Cov(aX, bY) = abCov(X, Y)$.

$\to Cov(X, a) = \mathbb{E}\left((X - \mathbb{E}X) \cdot \underbrace{(a - \mathbb{E}a)}_{=0}\right)$

$\to Cov(X+a, Y+b)$

$= \mathbb{E}\left(\underbrace{(X+a - \mathbb{E}(X+a))}_{= X - \mathbb{E}X} \cdot \underbrace{(Y+b - \mathbb{E}(Y+b))}_{= Y - \mathbb{E}Y}\right)$

Covariance

Properties:

(i) • $Cov(X, X) = Var(X) \geq 0$;

(ii) • $Cov(X, a) = 0, \quad a$ is a constant;

(iii) • $Cov(X, Y) = Cov(Y, X)$;

(iv) • $Cov(X + a, Y + b) = Cov(X, Y)$;

(v) • $Cov(aX, bY) = abCov(X, Y)$.

Corollary about variance:

$$Var(aX + b) = a^2 Var(X).$$

$$Var(aX+b) \overset{(i)}{=} Cov(aX+b, aX+b) \overset{(iv)}{=} Cov(aX, aX) \overset{(v)}{=} a^2 Cov(X, X)$$

$$= a^2 Var(X)$$

Covariance

Relate covariance to inner product:

Inner product (not rigorous)

Inner product is a operator from a vector space V to a field F (use $\mathbb{R}$ here as an example): $< \cdot, \cdot >: V \times V \to \mathbb{R}$ that satisfies:

- Symmetry: $< x, y > = < y, x >$;
- Linearity in the first argument: $< ax + by, z > = a < x, z > + b < y, z >$;
- Positive-definiteness: $< x, x > \geq 0$, and $< x, x > = 0 \Leftrightarrow x = 0$

$V =$ L^2 space: a space of square-integrable r.v.'s

$\mathbb{E} X^2 < \infty$

Covariance

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Remark:

Covariance defines an inner product over the quotient vector space obtained by taking the subspace of random variables with finite second moment and identifying any two that differ by a constant.

Covariance

Properties inherited from the inner product space

Recall in Euclidean vector space:

- $< x, y >= x^\top y = \sum_{i=1}^{n} x_i y_i$;
- $||x||_2 = \sqrt{< x, x >}$;
- $< x, y >= ||x||_2 \cdot ||y||_2 \cos(\theta)$.

Respectively:

- $< X, Y >= Cov(X, Y)$;
- $||X|| = \sqrt{Var(X)}$;

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Covariance

A substitute for $\cos(\theta)$**:**

Correlation

For two jointly distributed real-valued random variables X, Y with finite second moments, the correlation is defined as

$$Corr(X, Y) = \rho_{XY} = \frac{Cov(X, Y)}{\sqrt{Var(X) \cdot Var(Y)}}.$$

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Covariance

A substitute for $\cos(\theta)$:

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Uncorrelatedness:

$$X, Y \text{ uncorrelated} \quad \Leftrightarrow \quad Corr(X, Y) = 0.$$

Covariance

Cauchy-Schwarz inequality

$$|Cov(X,Y)| \le \sqrt{Var(X)Var(Y)}.$$

Proof: let $X - \mathbb{E}X = \tilde{X}$, $Y - \mathbb{E}Y = \tilde{Y}$.

$$0 \le \mathbb{E}(\tilde{X} + t\tilde{Y})^2 = \mathbb{E}\tilde{X}^2 + 2t\,\mathbb{E}(\tilde{X}\cdot\tilde{Y}) + t^2\,\mathbb{E}\tilde{Y}^2.$$

$t \in \mathbb{R}$ — quadratic w.r.t. t

Since the inequality holds for any $t \in \mathbb{R}$,

$$D/4 = Cov(X,Y)^2 - Var(X)Var(Y) \le 0$$

Covariance

Uncorrelatedness and Independence:

Observe the relationship:

$$Corr(X, Y) = 0 \quad \Leftrightarrow \quad Cov(X, Y) = 0 \quad \Leftrightarrow \quad \mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(\cancel{X}\,Y)$$

this happens
when X and Y
are independent

Covariance

Uncorrelatedness and Independence:

Observe the relationship:

$$Corr(X, Y) = 0 \quad \Leftrightarrow \quad Cov(X, Y) = 0 \quad \Leftrightarrow \quad \mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(X)$$

Conclusions:

- Independence $\Rightarrow$ Uncorrelatedness
- Uncorrelatedness $\not\Longrightarrow$ Independence

Remark:
Independence is a very strong assumption/property on the distribution.

Covariance

Special case: multivariate normal

Multivariate normal

A k-dimensional random vector $\mathbf{X} = (X_1, X_2, \cdots, X_k)^\top$ follows a multivariate normal distribution $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, if

$$f_{\mathbf{X}}(x_1, \ldots, x_k) = \frac{\exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)}{\sqrt{(2\pi)^k|\boldsymbol{\Sigma}|}},$$

where $\boldsymbol{\mu} = \mathbb{E}[\mathbf{X}] = (\mathbb{E}[X_1], \mathbb{E}[X_2], \ldots, \mathbb{E}[X_k])^\top$, and $[\boldsymbol{\Sigma}]_{i,j} = \Sigma_{i,j} = Cov(X_i, X_j)$.

mean vector of X

Σ covariance matrix, PSD

Observation:

The distribution is decided by the covariance structure.

$\Sigma = \mathbb{E}\,(X-\mu)\cdot(X-\mu)^T$

Since Σ is PSD, in particular symmetric,

$\exists$ orthogonal matrix V and diagonal $\Lambda = \begin{pmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_d^2 \end{pmatrix}$

$(V^T V = V V^T = I)$

s.t. $\underbrace{\Sigma = V \Lambda V^T \iff V^T \Sigma V = \Lambda}_{\text{spectral decomposition}}$.

$$\Sigma^{-1} = V \Lambda^{-1} V^T$$

$$(x-\mu)^T \Sigma^{-1} (x-\mu) = [\underbrace{V^T(x-\mu)}_{u}]^T \Lambda^{-1} [V^T(x-\mu)]$$

Let z

$$= z^T \Lambda^{-1} z = \sum \lambda_i^{-2} z_i^2 .$$

Also, $|\det \Sigma| = |\det \Lambda| = \prod_{i=1}^{n} \lambda_i^2$

Thus $\underbrace{f(x) = \prod_{i=1}^{n} \left\{ \frac{1}{\sqrt{2\pi}\,\lambda_i} \exp\left(- \frac{z_i^2}{2\lambda_i^2} \right) \right\}}$

$\to z_i$ are independent.

Covariance

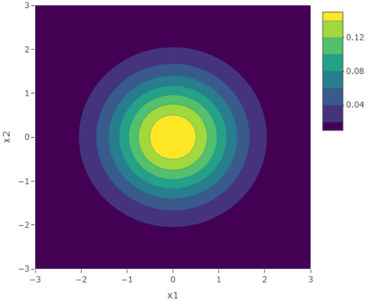
$$X_i, i = 1, \cdots k \text{ independent} \Leftrightarrow f_{\mathbf{X}}(x_1, \ldots, x_k) = \prod_{i=1}^{k} f_{X_i}(x_i)$$

$$\Leftrightarrow \boldsymbol{\Sigma} = I_k \Leftrightarrow Cov(X_i, X_j) = 0, i \neq j.$$

diagonal

Example:

- $Corr(X, Y) = 0$



Covariance

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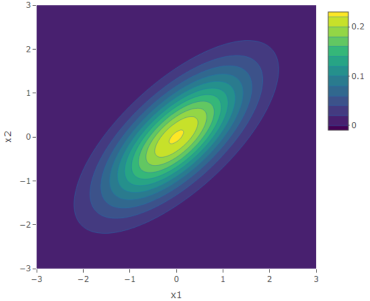
$$\Leftrightarrow \boldsymbol{\Sigma} = I_k \Leftrightarrow Cov(X_i, X_j) = 0, i \neq j.$$

Example:

- $Corr(X, Y) = 0.7$



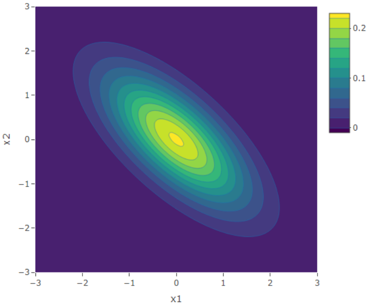
Covariance

$$X_i, i = 1, \cdots k \text{ independent} \Leftrightarrow f_{\mathbf{X}}(x_1, \ldots, x_k) = \prod_{i=1}^{k} f_{X_i}(x_i)$$

$$\Leftrightarrow \mathbf{\Sigma} = I_k \Leftrightarrow Cov(X_i, X_j) = 0, i \neq j.$$

Example:

- $Corr(X, Y) = -0.7$



Concentration

Measures of a distribution:

- $\mathbb{E}(X^k)$, $\mathbb{E}(X)$, $Var(X)$;
- $Cov(X, Y)$ and $Corr(X, Y)$.

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Concentration

Measures of a distribution:

- $\mathbb{E}(X^k)$, $\mathbb{E}(X)$, $Var(X)$;
- $Cov(X, Y)$ and $Corr(X, Y)$.

Tail probability: $\mathbf{P}(|X| > t)$

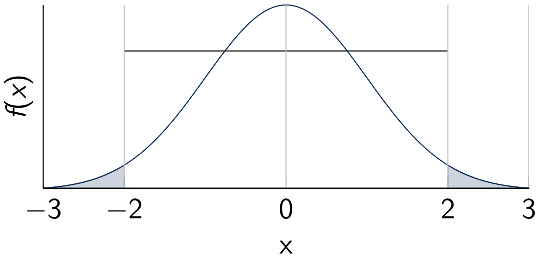

Figure: Probability density function of $\mathcal{N}(0, 1)$

Concentration

Concentration inequalities:

- Markov inequality
- Chebyshev inequality
- Chernoff bounds

Concentration

Concentration inequalities:

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Markov inequality

Let X be a random variable that is non-negative (almost surely). Then, for every constant $a > 0$,

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}.$$

Proof: We use monotonicity of Expectation, i.e.

if $X \geq Y$ then $\mathbb{E}X \geq \mathbb{E}Y$.

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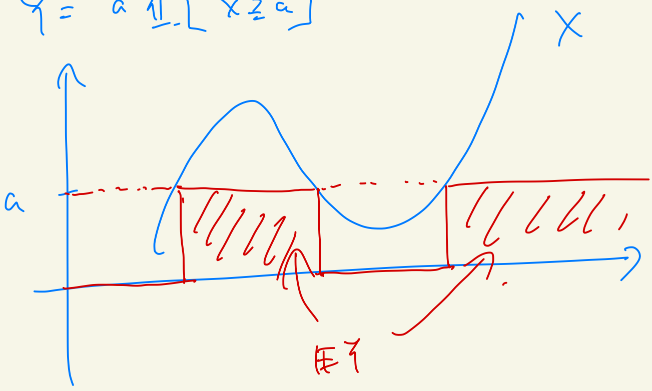
$Y = a\,\mathbb{1}[X \geq a]$



$\mathbb{E}X \geq \mathbb{E}Y = \mathbb{E}\, a\,\mathbb{1}[X \geq a]$) $\mathbb{P}(X \geq a) = \mathbb{E}\,\mathbb{1}[X \geq a]$

$= a\,\mathbb{P}(X \geq a)$

Concentration

Markov inequality (continued)

Let X be a random variable, then for every constant $a > 0$,

$$\mathbb{P}(|X| \geq a) \leq \frac{\mathbb{E}(|X|)}{a}.$$

A more general conclusion:

Markov inequality (continued)

Let X be a random variable, if $\Phi(x)$ is monotonically increasing on $[0, \infty)$, then for every constant $a > 0$,

by Markov

$$\mathbb{P}(|X| \geq a) = \mathbb{P}(\Phi(|X|) \geq \Phi(a)) \leq \frac{\mathbb{E}(\Phi(|X|))}{\Phi(a)}.$$

$\{|X| \geq a\} = \{\phi(|X|) \geq \phi(a)\}$

Concentration

By taking $\phi(x) = x^2$

Chebyshev inequality

Let X be a random variable with finite expectation $\mathbb{E}(X)$ and variance $Var(X)$, then for every constant $a > 0$,

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq a) \leq \frac{Var(X)}{a^2},$$

or equivalently,

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq a\sqrt{Var(X)}) \leq \frac{1}{a^2}.$$

Example:

Take $a = 2$,

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq 2\sqrt{Var(X)}) \leq \frac{1}{4}.$$

Concentration

Chernoff bound (general)

Let X be a random variable, then for $t \geq 0$,

by Markov.

$$\mathbb{P}(X \geq a) = \mathbb{P}(e^{t \cdot X} \geq e^{t \cdot a}) \leq \frac{\mathbb{E}\left[e^{t \cdot X}\right]}{e^{t \cdot a}},$$

$$\{x \geq a\} = \{t \cdot x \geq t \cdot a\} = \{\exp(t \cdot x) \geq \exp(t \cdot a)\}$$

and

$$\mathbb{P}(X \geq a) \leq \inf_{t \geq 0} \frac{\mathbb{E}\left[e^{t \cdot X}\right]}{e^{t \cdot a}}.$$

taking infimum of RHS above.

Remark:

This is especially useful when considering $X = \sum_{i=1}^{n} X_i$ with X_i's independent,

$$\mathbb{P}(X \geq a) \leq \inf_{t \geq 0} \frac{\mathbb{E}\left[\prod_i e^{t \cdot X_i}\right]}{e^{t \cdot a}} = \inf_{t \geq 0} e^{-t \cdot a} \prod_i \mathbb{E}\left[e^{t \cdot X_i}\right].$$

In particular, if $X_i \overset{\text{i.i.d.}}{\sim} Z$.

$$\mathbb{P}(X \geq a) \leq \inf_{t>0} \frac{\left(\mathbb{E}[e^{tZ}]\right)^n}{e^{ta}}.$$

e.g.) $X_i \overset{\text{i.i.d.}}{\sim} \text{Bern}(1/2)$

$$\mathbb{E}[e^{t \cdot X_i}] = \frac{e^t + e^{-t}}{2}$$

Thus $$\mathbb{P}\left(\sum_{i=1}^{n} X_i \geq a\right) \leq \inf_{t>0} \frac{\left(\frac{e^t + e^{-t}}{2}\right)^n}{e^{ta}}.$$

Problem Set

Problem 1: Let

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \leq y \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases},$$

compute $Cov(X, Y)$.

Problem 2: For $X \sim \mathcal{N}(0, 1)$, compute the Chernoff bound.