UNIVERSITY OF TORONTO | Statistical Sciences

DoSS Summer Bootcamp Probability Module 9

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Outline

- Counterexamples

Counterexamples

Recall: A random variable $X \in L^p$ if $\|X\|_{L^p} = (E|X|^p)^{1/p} < \infty$.

$X_n \to X$ in L^p if $\lim_{n\to\infty} \|X_n - X\|_{L^p} = 0$

Monotonicity of L^p Convergence

If $q > p > 0$, L^q convergence implies L^p convergence

Counterexample to the Converse:

Let $\mathbb{P}(X_n = n^a) = \frac{1}{n} = 1 - \mathbb{P}(X_n = 0)$.

$\mathbb{E}|X_n|^p = \frac{1}{n} \cdot n^{ap} = n^{ap-1}$, $\mathbb{E}|X_n|^q = n^{aq-1}$

Pick a so that $ap - 1 < 0 < aq - 1$.

This is always possible since $q > p$.

Then, $\lim_{n\to\infty} \mathbb{E}|X_n|^p = 0 \neq \infty = \lim_{n\to\infty} \mathbb{E}|X_n|^q$

that meas $X_n \to 0$ in L^p but $X_n \not\to 0$ in L^q.

Counterexamples

Recall: X_n converges to X in probability if for any $\epsilon > 0$ $\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0$.

L^p convergence implies Convergence in Probability

If $X_n \to X$ in L^p, then $X_n \to X$ in probability.

Counterexample to the Converse:

Let $\mathbb{P}(X_n = n^{1/p}) = \frac{1}{n} = 1 - \mathbb{P}(X_n = 0)$

For any $\varepsilon > 0$, $\mathbb{P}(|X_n - 0| > \varepsilon) = \mathbb{P}(X_n = n^{1/p}) = \frac{1}{n} \to 0$ as $n \to \infty$

for sufficiently large n.

So $X_n \to 0$ in probability

$$\mathbb{E}|X_n|^p = \frac{1}{n} \cdot (n^{1/p})^p = 1 \not\to 0.$$

Thus X_n does not converge to 0 in L^p.

Counterexamples

Recall: X_n converges to X in probability if for any $\epsilon > 0$ $\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0$.

a.s. Convergence implies Convergence in Probability

If $X_n \to X$ almost surely, then $X_n \to X$ in probability.

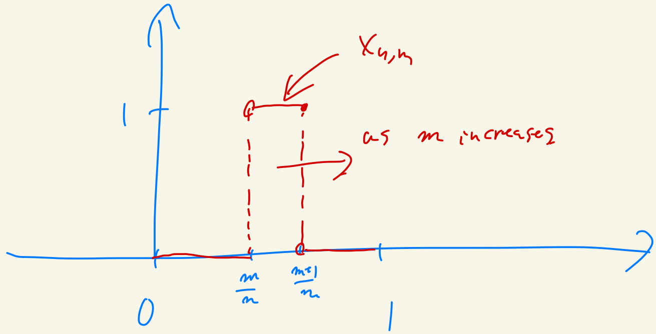
Counterexample to the Converse:

Let $\Omega = (0,1)$, $\mathcal{F} = \mathcal{B}(\Omega)$ Borel sets on $(0,1)$,

$\mathbb{P}$ on Ω be uniform measure, i.e. $\mathbb{P}(\omega \in (a,b)) = b-a$

if $0 < a < b < 1$.

Define $X_{n,m}(\omega) = \begin{cases} 1 & \text{if } \omega \in [\frac{m}{n}, \frac{m+1}{n}] \quad 0 \leq m \leq n-1 \\ 0 & \text{otherwise.} \end{cases}$



This makes $\mathbb{P}(X_{n,m}=1) = \frac{1}{n} = 1 - \mathbb{P}(X_{n,m}=0)$

So, $\mathbb{P}(|X_{n,m}| > \varepsilon) \xrightarrow{\le \frac{1}{n}} 0$ as $n \to \infty$.

Thus $X_{n,m} \to 0$ in probability.

But for each $\omega \in (0,1)$

Then by the construction,

$X_{n,m}(\omega) = 1$ for infinitely many n, m

(For each n, you can find m s.t. $\omega \in (\frac{m}{n}, \frac{m+1}{n}]$)

Thus $X_{n,m} \not\to 0$ for any point on Ω.

Counterexamples

Recall: X_n converges to X in distribution if for any continuity point x of $P(X \le x)$, $\lim_{n\to\infty} P(X_n \le x) = P(X \le x)$ holds.

Convergence in Probability implies Convergence in Distribution

If $X_n \to X$ in probability, then $X_n \to X$ in distribution.

Counterexample to the Converse:

Let $X \sim$ symmetric Bernoulli $(\frac{1}{2})$ i.e. $\mathbb{P}(X = 1) = \frac{1}{2} = \mathbb{P}(X = -1)$

Define $X_n = (-1)^n X$.

Since X is symmetric, $X_n =_d X$ for any n.

Therefore, $X_n \to X$ in distribution.

However, for any odd n,

$$\mathbb{P}\left(|X_n - X| > 1 \right) = 1 \quad \text{since } X_n = -X \text{ and } |X| = 1.$$

Therefore $X_n \not\to X$ in probability.

Counterexamples

Recall: X_n converges to X in distribution if for any continuity point x of $P(X \leq x)$, $\lim_{n\to\infty} P(X_n \leq x) = P(X \leq x)$ holds.

Convergence in Probability implies Convergence in Distribution

If $X_n \to X$ in probability, then $X_n \to X$ in distribution.

Special case when the Converse holds:

we discussed this slide before.

Counterexamples

Monotone Convergence Theorem

If $X_n \geq 0$ and $X_n \nearrow X$, then $EX_n \nearrow EX$

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Counterexample when X_n is not lower bounded:

Let X be $\mathbb{P}(X = -2^i) = 2^{-i}$ for $i = 1, 2, 3, \ldots\ldots$

then $\mathbb{P}(X < 0) = 1$, i.e. $X < 0$ a.s.

Let $X_n = \dfrac{X}{n}$

Since $X < 0$ a.s. $X_n \nearrow 0$.

However, X_n is not lower bounded

since $-2^i \to -\infty$ as $i \to \infty$.

Apparently, $\mathbb{E}\, 0 = 0$.

However $\mathbb{E} X_n = \frac{1}{n} \cdot \mathbb{E} X$

$$= \frac{1}{n} \cdot \underbrace{\sum_{i=1}^{\infty} 2^{-i} \cdot (-2^{i})}_{= -\infty} = -\infty$$

Thus this example shows $X_n \nearrow 0$

but $\mathbb{E} X_n \not\to \mathbb{E} X$.

Counterexamples

Dominated Convergence Theorem

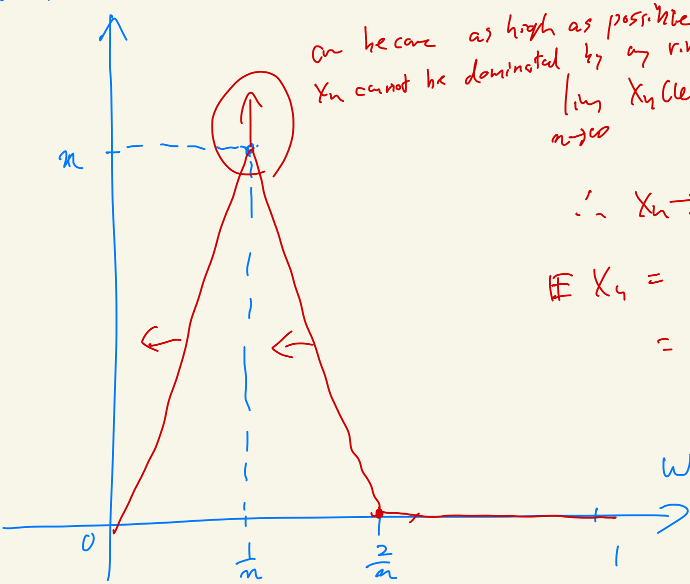
If $X_n \to X$ a.s. and $|X_n| \leq Y$ a.s. for all n and Y is integrable, then $EX_n \to EX$

Counterexample when X_n is not dominated by an integrable random variable:

Let $\Omega = (0,1)$ with uniform distribution $\mathbb{P}$.

i.e. $\mathbb{P}(\omega \in (a,b)) = b-a$ if $0 < a < b < 1$

Then define $X_n(\omega)$ as follows



can be come as high as possible.

X_n cannot be dominated by any r.v. Y.

$\lim_{n\to\infty} X_n(\omega) = 0$ for any $\omega \in (0,1)$

$\therefore X_n \to 0$ a.s.

$\mathbb{E} X_n =$ area of the triangle

$= 1 \not\to 0.$