Problem 1

Prove that for a σ-field $\mathcal{F}$, if $A_1, A_2, \cdots \in \mathcal{F}$, then $\cap_{i=1}^{\infty} A_i \in \mathcal{F}$.

Solution:

Proof. By the properties of σ-field, $A_1^c, A_2^c, \cdots \in \mathcal{F}$, and

$$\cup_{i=1}^{\infty} A_i^c \in \mathcal{F},$$

then by the first property,

$$(\cup_{i=1}^{\infty} A_i^c)^c = \cap_{i=1}^{\infty} A_i \in \mathcal{F}.$$

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Problem 2

Prove monotonicity and subadditivity of measure μ on σ-field.

Solution:

Proof.

- If $A \subseteq B$, then $A \cup (B \cap A^c) = B$, and $A \cap (B \cap A^c) = \varnothing$. Therefore,

$$\mu(B) = \mu(A) + \mu((B \cap A^c)) \geq \mu(A).$$

- If $A \subseteq \cup_{i=1}^{\infty} A_i$, consider $A_i' = A \cap A_i$, and let

$$B_1 = A_1', \quad B_i = A_i' \cap (\cup_{k=1}^{i-1} A_k')^c,$$

then $B_i, i \geq 1$ are disjoint, $B_i \subseteq A_i' \subseteq A_i$, and $\cup_{i=1}^{\infty} B_i = \cup_{i=1}^{\infty} A_i' = A$. Therefore,

$$\mu(A) = \mu(\cup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \mu(B_i) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

∎

Problem 3

(Monty Hall problem) Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?
(Assumptions: the host will not open the door we picked and the host will only open the door which has a goat.)

Solution:

Proof. Denote the event "The car is behind Door i" as D_i, and denote the event "The host opens Door i" as O_i, then

$$\mathbb{P}(D_i) = \frac{1}{3},$$

and

$$\mathbb{P}(O_3 \mid D_1) = \frac{1}{2}, \quad \mathbb{P}(O_3 \mid D_2) = 1, \quad \mathbb{P}(O_3 \mid D_3) = 0.$$

Therefore,

$$\begin{aligned}\mathbb{P}(D_2 \mid O_3) &= \frac{\mathbb{P}(O_3 \mid D_2)\mathbb{P}(D_2)}{\mathbb{P}(O_3 \mid D_1)\mathbb{P}(D_1) + \mathbb{P}(O_3 \mid D_2)\mathbb{P}(D_2) + \mathbb{P}(O_3 \mid D_3)\mathbb{P}(D_3)} \\ &= \frac{\frac{1}{3}}{\frac{1}{6} + \frac{1}{3} + 0} = \frac{2}{3} > \mathbb{P}(D_1 \mid O_3),\end{aligned}$$

which means we should switch. ∎