Problem 1

Give an example where the events are pairwise independent but not mutually independent.

Solution:

Proof. Omitted. ■

Problem 2

Verify that the measure $\mu(\cdot)$ induced by $P(\cdot)$ is a probability measure on $\mathcal{R}$.

Solution:

Proof.

- $\mu(\varnothing) = \mathbb{P}(X \in \varnothing) = \mathbb{P}(\varnothing) = 0$;
- $\mu(\mathbb{R}) = \mathbb{P}(X \in \mathbb{R}) = \mathbb{P}(\Omega) = 1$;
- If $A_1, A_2, \dots \in \mathcal{R}$ are disjoint, then $X^{-1}(A_1), X^{-1}(A_2), \dots \in \mathcal{F}$ are also disjoint,

$$\mu(\cup_{i=1}^{\infty} A_i) = \mathbb{P}(X \in \cup_{i=1}^{\infty} A_i) = \mathbb{P}(\cup_{i=1}^{\infty} X^{-1}(A_i)) = \sum_{i=1}^{\infty} \mathbb{P}(X^{-1}(A_i)) = \sum_{i=1}^{\infty} \mu(A_i).$$

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Problem 3

Prove properties 3 - 5 of CDF $F(\cdot)$.

Solution:

Proof.

- Right continuity: using continuity from above.
  Reference: A proof from Proof Wiki.
- Similar to the proof of right continuity, for this we use continuity from below, the proof is completed by observing that for any increasing $\{x_n\}_{n=1}^{\infty}$ with $x_n < x$ and $x_n \to x$, the limit of $(-\infty, x_n]$ is

$$\cup_{n=1}^{\infty} (-\infty, x_n] = (-\infty, x).$$

- By the aforementioned 2 results,

$$\mathbb{P}(X = x) = \mathbb{P}(X \leq x) - \mathbb{P}(X < x) = F(x) - F(x^-)$$

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Problem 4

Bob and Alice are playing a game. They alternatively keep tossing a fair coin and the first one to get a H wins. Does the person who plays first have a better chance at winning?

Solution:

Proof. Without loss of generality, suppose Alice plays first. Then denote A as the event "Alice wins", we have

$$\mathbb{P}(A) = \mathbb{P}(\{H\}, \{TTH\}, \{TTTTH\}, \cdots) = \sum_{i=1}^{\infty} (\frac{1}{2})^{2i-1} = \frac{2}{3} > 1 - \mathbb{P}(A),$$

so the person who plays first have a better chance at winning. ∎