Problem 1

Let
$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases},$$
compute $Cov(X,Y)$.

Solution:

Proof. First compute the marginal density of X and Y by taking the integration, we have
$$f_X(x) = 2x, \quad 0 \le x \le 1,$$
$$f_Y(y) = 2(1-y), \quad 0 \le y \le 1.$$
■

Then $\mathbb{E}(X) = \frac{2}{3}$, $\mathbb{E}(Y) = \frac{1}{3}$, and
$$\begin{aligned} \mathbb{E}(XY) &= \int_{\mathbb{R}^2} xy f_{X,Y}(x,y)\, dxdy \\ &= \int_0^1 (\int_0^x 2xy\, dy)\, dx \\ &= \frac{1}{4}. \end{aligned}$$
Therefore, $Cov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = \frac{1}{36}$.

Problem 2

For $X \sim \mathcal{N}(0,1)$, compute the Chernoff bound.

Solution:

Proof. Based on Problem 3 in Module 5, the MGF of X is
$$M_X(t) = \mathbb{E}(e^{tX}) = e^{\frac{t^2}{2}}.$$
Therefore, for $t \ge 0$, the Chernoff bound is
$$\mathbb{P}(X \ge a) \le \inf_{t\ge 0} \frac{M_X(t)}{e^{ta}} = \inf_{t \ge 0} e^{\frac{t^2}{2} - at} = e^{-\frac{a^2}{2}} \cdot \inf_{t\ge 0} e^{\frac{1}{2}(t-a)^2}$$
Specifically, for $a > 0$,
$$\mathbb{P}(X \ge a) \le e^{-\frac{a^2}{2}},$$
and by symmetry,
$$\mathbb{P}(|X| \ge a) \le 2e^{-\frac{a^2}{2}}.$$
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