Problem 1

Prove that on a complete probability space, if $X_n \xrightarrow{a.s.} X, Y_n \xrightarrow{a.s.} Y$, then $X_n + Y_n \xrightarrow{a.s.} X + Y$.

Solution:

Proof. Denote $\Omega_1 = \{\omega : \lim_{n\to\infty} X_n(\omega) \neq X(\omega)\}$, and $\Omega_2 = \{\omega : \lim_{n\to\infty} Y_n(\omega) \neq Y(\omega)\}$, then by almost sure convergence, we have

$$\mathbb{P}(\Omega_1) = \mathbb{P}(\Omega_2) = 0.$$

Considering $\omega \in \Omega_1^c \cap \Omega_2^c$, we must have $\lim_{n\to\infty} X_n(\omega) + Y_n(\omega) = X(\omega) + Y(\omega)$, so

$$\begin{aligned}
\mathbb{P}(\{\omega : \lim_{n\to\infty} X_n(\omega) + Y_n(\omega) \neq X(\omega) + Y(\omega)\}) &= 1 - \mathbb{P}(\{\omega : \lim_{n\to\infty} X_n(\omega) + Y_n(\omega) = X(\omega) + Y(\omega)\}) \\
&\leq 1 - \mathbb{P}(\Omega_1^c \cap \Omega_2^c) \\
&= \mathbb{P}(\Omega_1 \cup \Omega_2) \\
&\leq \mathbb{P}(\Omega_1) + \mathbb{P}(\Omega_2) \\
&= 0
\end{aligned}$$

Thus, $\mathbb{P}(\{\omega : \lim_{n\to\infty} X_n(\omega) + Y_n(\omega) \neq X(\omega) + Y(\omega)\}) = 0$. The conclusion follows. ■

Problem 2

Prove that on a complete probability space, if $X_n \xrightarrow{P} X, Y_n \xrightarrow{P} Y$, then $X_n + Y_n \xrightarrow{P} X + Y$.

Solution:

Proof. For $\epsilon > 0$, consider

$$\begin{aligned}
P\left(|X_n + Y_n - (X + Y)| > \varepsilon\right) &\leq P\left(|X - X_n| + |Y - Y_n| > \varepsilon\right) \\
&\leq P\left(|X - X_n| > \varepsilon/2\right) + P\left(|Y - Y_n| > \varepsilon/2\right),
\end{aligned}$$

the result follows immediately by the convergence of probability of X_n and Y_n. ■

Problem 3

A bank teller serves customers standing in the queue one by one. Suppose that the service time X_i for customer i has mean $\mathbb{E}(X_i) = 2$ (minutes) and $Var(X_i) = 1$. We assume that service times for different bank customers are independent. Let Y be the total time the bank teller spends serving 50 customers. Find $\mathbb{P}(90 < Y < 110)$.

Solution:

Proof. Since $Y = \sum_{i=1}^{50} X_i$, $n = 50$ is considered to be sufficiently large, then by CLT,

$$\frac{Y - 100}{\sqrt{50}} \xrightarrow{d} Z = \mathcal{N}(0, 1), \text{approximately.}$$

Therefore

$$\begin{aligned} P(90 < Y < 110) &= P\left(\frac{90 - 100}{\sqrt{50}} < \frac{Y - 100}{\sqrt{50}} < \frac{110 - 100}{\sqrt{50}}\right) \\ &\approx P\left(-\sqrt{2} < Z < \sqrt{2}\right) \\ &= 0.8427. \end{aligned}$$

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