UNIVERSITY OF TORONTO | Statistical Sciences

# DoSS Summer Bootcamp Probability
# Module 1

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# Roadmap

**A bridge connecting undergraduate probability and graduate probability**

**Undergraduate-level probability**

- Concrete;
- Examples and scenarios;
- Rely on computation...

# Roadmap

## A bridge connecting undergraduate probability and graduate probability

### Undergraduate-level probability

- Concrete;
- Examples and scenarios;
- Rely on computation...

### Graduate-level probability

- Abstract (measure theory);
- Laws and properties;
- Rely on construction and inference...

# Roadmap

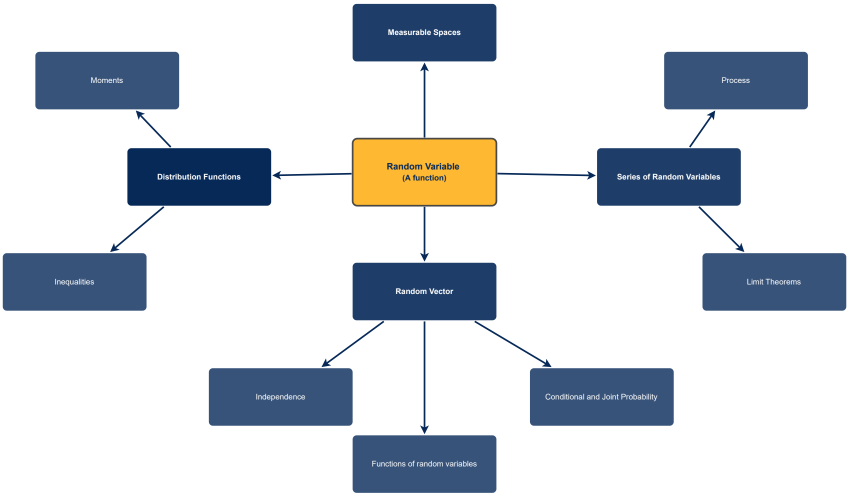


Figure: Roadmap

# Outline

- Measurable spaces
  - Sample Space
  - $\sigma$-algebra
- Probability measures
  - Measures on $\sigma$-field
  - Basic results
- Conditional probability
  - Bayes' rule
  - Law of total probability

# Measurable spaces

**Sample Space**

The sample space $\Omega$ is the set of all possible outcomes of an experiment.

**Examples:**

- Toss a coin: $\{H, T\}$
- Roll a die: $\{1, 2, 3, 4, 5, 6\}$

# Measurable spaces

**Sample Space**

The sample space $\Omega$ is the set of all possible outcomes of an experiment.

**Examples:**

- Toss a coin: $\{H, T\}$
- Roll a die: $\{1, 2, 3, 4, 5, 6\}$

**Event**

An event is a collection of possible outcomes (subset of the sample space).

**Examples:**

- Get head when tossing a coin: $\{H\}$
- Get an even number when rolling a die: $\{2, 4, 6\}$

# Measurable spaces

**$\sigma$-algebra**

A $\sigma$-algebra ($\sigma$-field) $\mathcal{F}$ on $\Omega$ is a non-empty collection of subsets of $\Omega$ such that

- If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$,
- If $A_1, A_2, \cdots \in \mathcal{F}$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

**Remark:** $\varnothing, \Omega \in \mathcal{F}$

# Probability measures

**Measures on $\sigma$-field**

A function $\mu : \mathcal{F} \to R^+ \cup \{+\infty\}$ is called a measure if

- $\mu(\varnothing) = 0$,
- If $A_1, A_2, \dots \in \mathcal{F}$ and $A_i \cap A_j = \varnothing$, then $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.

If $\mu(\Omega) = 1$, then $\mu$ is called a probability measure.

# Probability measures

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**Properties:**

- Monotonicity: $A \subseteq B \quad \Rightarrow \quad \mu(A) \leq \mu(B)$
- Subadditivity: $A \subseteq \cup_{i=1}^{\infty} A_i \quad \Rightarrow \quad \mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$
- Continuity from below: $A_i \nearrow A \quad \Rightarrow \quad \mu(A_i) \nearrow \mu(A)$
- Continuity from above: $A_i \searrow A$ and $\mu(A_i) < \infty \quad \Rightarrow \quad \mu(A_i) \searrow \mu(A)$

# Probability measures

**Proof of continuity from below:**

# Probability measures

**Proof of continuity from above:**

**Remark:** $\mu(A_i) < \infty$ is vital.

# Probability measures

**Examples:**

$\Omega = \{\omega_1, \omega_2, \cdots\}$, $A = \{\omega_{a_1}, \cdots, \omega_{a_i}, \cdots\} \Rightarrow \mu(A) = \sum_{j=1}^{\infty} \mu(\omega_{a_j})$.
Therefore, we only need to define $\mu(\omega_j) = p_j \geq 0$.
If further $\sum_{i=1}^{\infty} p_j = 1$, then $\mu$ is a probability measure.

- Toss a coin:

- Roll a die:

# Conditional probability

**Original problem**:

- What is the probability of some event $A$?
- $P(A)$ is determined by our probability measure.

**New problem**:

- Given that $B$ happens, what is the probability of some event $A$?
- $P(A \mid B)$ is the conditional probability of the event $A$ given $B$.

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- $P(A \mid B)$ is the conditional probability of the event $A$ given $B$.

**Example:**

- Roll a die: $P(\{2\} \mid \text{even number})$

# Conditional probability

**Bayes' rule**

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

**Remark:** Does conditional probability $P(\cdot \mid B)$ satisfy the axioms of a probability measure?

# Conditional probability

### Multiplication rule

$$P(A \cap B) = P(A \mid B)P(B) = P(B \mid A)P(A)$$

**Generalization**:

### Law of total probability

Let $A_1, A_2, \cdots, A_n$ be a partition of $\omega$, such that $P(A_i) > 0$, then

$$P(B) = \sum_{i=1}^{n} P(A_i)P(B \mid A_i)$$

# Problem Set

**Problem 1:** Prove that for a $\sigma$-field $\mathcal{F}$, if $A_1, A_2, \cdots \in \mathcal{F}$, then $\cap_{i=1}^{\infty} A_i \in \mathcal{F}$.

**Problem 2:** Prove monotonicity and subadditivity of measure $\mu$ on $\sigma$-field.

**Problem 3:** (Monty Hall problem) Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?
(Assumptions: the host will not open the door we picked and the host will only open the door which has a goat.)