UNIVERSITY OF TORONTO | Statistical Sciences

DoSS Summer Bootcamp Probability Module 1

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July 8, 2025

Roadmap

A bridge connecting undergraduate probability and graduate probability

Undergraduate-level probability

- Concrete;
- Examples and scenarios;
- Rely on computation...

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A bridge connecting undergraduate probability and graduate probability

Undergraduate-level probability

- Concrete;
- Examples and scenarios;
- Rely on computation...

Graduate-level probability

- Abstract (measure theory);
- Laws and properties;
- Rely on construction and inference...

Roadmap

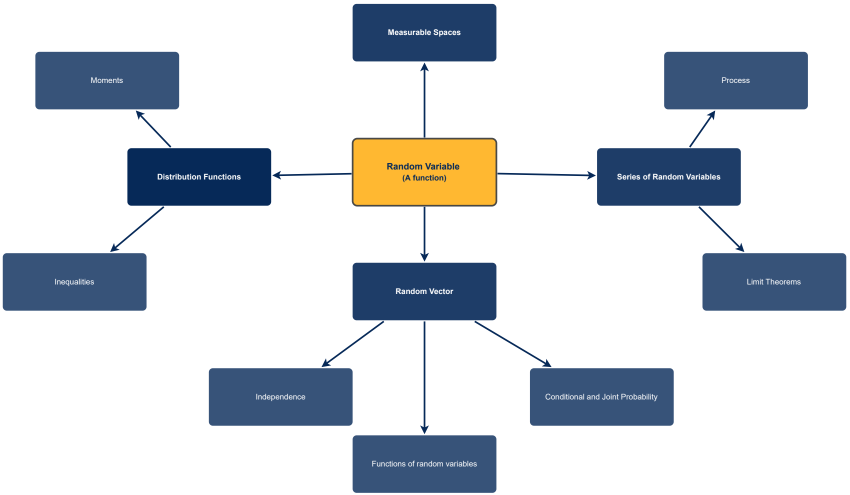


Figure: Roadmap

Outline

- Measurable spaces
  - Sample Space
  - σ-algebra
- Probability measures
  - Measures on σ-field
  - Basic results
- Conditional probability
  - Bayes' rule
  - Law of total probability

Today

Tomorrow?

Measurable spaces

Sample Space

The sample space Ω is the set of all possible outcomes of an experiment.

Examples:

- Toss a coin: $\{H, T\}$ $= \Omega$
- Roll a die: $\{1, 2, 3, 4, 5, 6\}$ $= \Omega$

Measurable spaces

Sample Space

The sample space Ω is the set of all possible outcomes of an experiment.

Examples:

- Toss a coin: $\{H, T\}$ $= \Omega$
- Roll a die: $\{1, 2, 3, 4, 5, 6\}$ $= \Omega$

Event

An event is a collection of possible outcomes (subset of the sample space).

Examples:

- Get head when tossing a coin: $\{H\}$ $\subset \Omega$
- Get an even number when rolling a die: $\{2, 4, 6\}$ $\subset \Omega$

What is the motivation for developing measure theory?

1) Observation from simple examples

Tossing a coin twice (discrete)

$\Omega = \{HH, HT, TH, TT\}$, $\rightarrow$ discrete.

$$\mathbb{P}(HH) = \mathbb{P}(HT) = \mathbb{P}(TH) = \mathbb{P}(TT) = \frac{1}{4}$$

Let X = the number of H.

$$\mathbb{P}(X=0) = \mathbb{P}(X=2) = \frac{1}{4}, \quad \mathbb{P}(X=1) = \frac{1}{2}$$

$$\mathbb{P}(X=0) + \mathbb{P}(X=1) + \mathbb{P}(X=2) = 1$$

$$\mathbb{E}X = \frac{1}{4}\cdot 0 + \frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2 = 1$$

Gaussian (continuous)

Let $X \sim \mathcal{N}(\mu, \sigma^2)$

Density $\quad p(x) = \dfrac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\dfrac{(x-\mu)^2}{2\sigma^2}\right)$

$$\int_{-\infty}^{\infty} p(x)\, dx = 1$$

$$\mathbb{E}X = \int_{-\infty}^{\infty} x\,p(x)\, dx = \mu$$

Discrete

$$\mathbb{P}(X \leq x) = \sum_{\ell \leq x} \mathbb{P}(X = \ell)$$

$$\mathbb{E}X = \sum_{k=-\infty}^{\infty} k\, \mathbb{P}(X = k)$$

Continuous

$$\mathbb{P}(X \leq x) = \int_{-\infty}^{x} p(x)\, dx$$

$$\mathbb{E}X = \int_{-\infty}^{\infty} x\, p(x)\, dx$$

Q. Is there any way to explain both in a unified manner?

2) Further Observation

If $A \cap B = \phi$, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ --- (*)

For a discrete case, $\{X = k\}$ are disjoint.

Repeating (*),

$$1 = \mathbb{P}(\Omega) = \underbrace{\sum_{k=-\infty}^{\infty} \mathbb{P}(X = k)}_{\text{countable sum}}$$

But for continuous case.

$$\mathbb{P}(X = x) = 0 \quad \text{for any } x \in \mathbb{R}.$$

uncountable sum cannot be defined well.

Therefore,

$$1 = \sum_{x \in \mathbb{R}} \mathbb{P}(X = x) = \sum_{x \in \mathbb{R}} 0 = 0$$

contradiction?

$\Longrightarrow$ uncountable sum is problematic!

Measurable spaces

σ-algebra

A σ-algebra (σ-field) $\mathcal{F}$ on Ω is a non-empty collection of subsets of Ω such that

(i) • If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$, $\longrightarrow$ complement is also in $\mathcal{F}$

(ii) • If $A_1, A_2, \dots \in \mathcal{F}$, then $\underbrace{\cup_{i=1}^{\infty} A_i}_{\text{countable union}} \in \mathcal{F}$. $\longrightarrow$ countable union is also in $\mathcal{F}$

Remark: $\emptyset, \Omega \in \mathcal{F}$

(Pf) Let $A \in \mathcal{F}$.

By (i), $A^c \in \mathcal{F}$.

By (ii), $\Omega = A \cup A^c \in \mathcal{F}$

Again by (i), $\emptyset = \Omega^c \in \mathcal{F}$ //

Remark

If $A_1, A_2, \cdots \in \mathcal{F}$, then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$.

(pf)

$$\left(\bigcap_{i=1}^{\infty} A_i\right)^c = \bigcup_{i=1}^{\infty} \underbrace{A_i^c}_{\substack{\cap \\ \text{(i)}\ \mathcal{F}}} \overset{\text{(ii)}}{\in} \mathcal{F}$$

Then by taking the complement of above,

we have $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$.

Construction of Probability Theory

Outline

1) Define the collection of subsets of Ω, $\mathcal{F}$ (σ-algebra) on which we define "Probability measure".

2) Define as a function

$$\mathbb{P} : \mathcal{F} \to [0, 1]$$

which has "countable additivity".

3) $(\Omega, \mathcal{F}, \mathbb{P})$ is called "Probability triple".

Ω: sample space. $\mathcal{F}$: σ-algebra. $\mathbb{P}$: probability measure.

Probability measures

Measures on σ-field

A function $\mu : \mathcal{F} \to R^+ \cup \{+\infty\}$ is called a measure if

- $\mu(\varnothing) = 0$,
- If $A_1, A_2, \cdots \in \mathcal{F}$ and $A_i \cap A_j = \varnothing$, then $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.

If $\mu(\Omega) = 1$, then μ is called a probability measure.

contable additivity

Probability measures

Measures on σ-field

A function $\mu : \mathcal{F} \to R^+ \cup \{+\infty\}$ is called a measure if

A_i must be disjoint

- $\mu(\varnothing) = 0$,
- If $A_1, A_2, \cdots \in \mathcal{F}$ and $A_i \cap A_j = \varnothing$, then $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.

If $\mu(\Omega) = 1$, then μ is called a probability measure.

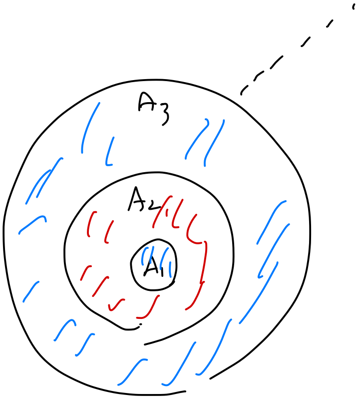
Properties:

- Monotonicity: $A \subseteq B \quad \Rightarrow \quad \mu(A) \leq \mu(B)$
- Subadditivity: $A \subseteq \cup_{i=1}^{\infty} A_i \quad \Rightarrow \quad \mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$ union bound in probability
- Continuity from below: $A_i \nearrow A \quad \Rightarrow \quad \mu(A_i) \nearrow \mu(A)$
- Continuity from above: $A_i \searrow A$ and $\mu(A_i) < \infty \quad \Rightarrow \quad \mu(A_i) \searrow \mu(A)$

Probability measures

Proof of continuity from below: $A_i \nearrow A$

$\iff A_1 \subset A_2 \subset A_3 \subset \cdots$, $\bigcup_{i=1}^{\infty} A_i = A$



Let $B_i = A_i \setminus A_{i-1}$, $i \geq 2$, $B_1 = A_1$.

Then B_i's are disjoint but $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i = A$.

By countable additivity

$$\mu(A) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu(B_i)$$

$$= \sum_{i=2}^{\infty} \left\{ \mu(A_i) - \mu(A_{i-1}) \right\} + \mu(A_1)$$

$$= \cancel{\sum_{i=1}^{\infty}} \mu(A_i)$$

$$\lim_{i\to\infty}$$

$$= \lim_{n\to\infty} \left\{ \sum_{i=2}^{n} \left\{ \mu(A_i) - \mu(A_{i-1}) \right\} + \mu(A_1) \right\}$$

$$= \lim_{n\to\infty} \left\{ \cancel{\mu(A_1)} + \left(\cancel{\mu(A_2)} - \cancel{\mu(A_1)} \right) + \left(\cancel{\mu(A_3)} - \cancel{\mu(A_2)} \right) + \cdots + \left(\underbrace{\mu(A_n)}_{\text{remain}} - \cancel{\mu(A_{n-1})} \right) \right\}$$

$$= \lim_{n\to\infty} \mu(A_n)$$

Probability measures

Proof of continuity from above:

$A_i \searrow A \iff A_1 \supset A_2 \supset A_3 \supset \cdots\cdots, \bigcap_{i=1}^{\infty} A_i = A.$

Let $B_i = A_1 - A_i$, we have $B_i \nearrow A_1 \setminus A$.

By the continuity from below

$$\lim_{i\to\infty} \mu(B_i) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu(A_1 \setminus A).$$

Since $\mu(A_1 \setminus A) = \mu(A_1) - \mu(A)$ and

$$\mu(B_i) = \mu(A_1 \setminus A_i) = \mu(A_1) - \mu(A_i),$$

$$\mu(A_1) - \lim_{i\to\infty} \mu(A_i) = \mu(A_1) - \mu(A)$$

$$\therefore \lim_{i\to\infty} \mu(A_i) = \mu(A).$$

Remark: $\mu(A_i) < \infty$ is vital.

Probability measures

countable additivity.

Examples:

discrete case.

$\Omega = \{\omega_1, \omega_2, \cdots\}$, $A = \{\omega_{a_1}, \cdots, \omega_{a_i}, \cdots\} \Rightarrow \mu(A) = \sum_{j=1}^{\infty} \mu(\omega_{a_j})$.
Therefore, we only need to define $\mu(\omega_j) = p_j \geq 0$.
If further $\sum_{i=1}^{\infty} p_j = 1$, then μ is a probability measure.

- Toss a coin:

  Define $\mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2}$. Then $\mathbb{P}$ is a probability measure.

- Roll a die:

  $\mathbb{P}(1) = \mathbb{P}(2) = \cdots = \mathbb{P}(6) = \frac{1}{6}$

  Then $\mathbb{P}$ is a probability measure.

Q. How does $(\Omega, \mathcal{F}, \mathbb{P})$ provide a unified theory?

sample space — Ω; σ-algebra — $\mathcal{F}$; probability measure — $\mathbb{P}$.

Observation

(Discrete r.v. X)

$$1 = \mathbb{P}(\Omega) = \sum_{k=-\infty}^{\infty} \mathbb{P}(X=k)$$

$$\mathbb{E}X = \sum_{k=-\infty}^{\infty} k\,\mathbb{P}(X=k)$$

(Continuous r.v.)

$$1 = \mathbb{P}(\Omega) = \int_{-\infty}^{\infty} p(x)\,dx$$

$$\mathbb{E}X = \int_{-\infty}^{\infty} x\,p(x)\,dx.$$

Consider Approximation

$$\Omega = \bigcup_{i=-\infty}^{\infty} \left\{ X \in \left[\frac{i}{n}, \frac{i+1}{n}\right) \right\}$$ (becomes finer as $n \nearrow \infty$)

$$1 = \mathbb{P}(\Omega) = \sum_{i=-\infty}^{\infty} \mathbb{P}\left(X \in \left[\frac{i}{n}, \frac{i+1}{n}\right) \right)$$

Approximation of Expectation

$$\mathbb{E}X \approx \sum_{i=-\infty}^{\infty} \frac{i}{n} \mathbb{P}\left(X \in \left[\frac{i}{n}, \frac{i+1}{n}\right)\right)$$

should become precise as $n \to \infty$

$\Downarrow$

We can instead define $\mathbb{E}X$ as the limit of the approximation above.

(informal measure-theoretic definition of expectation)

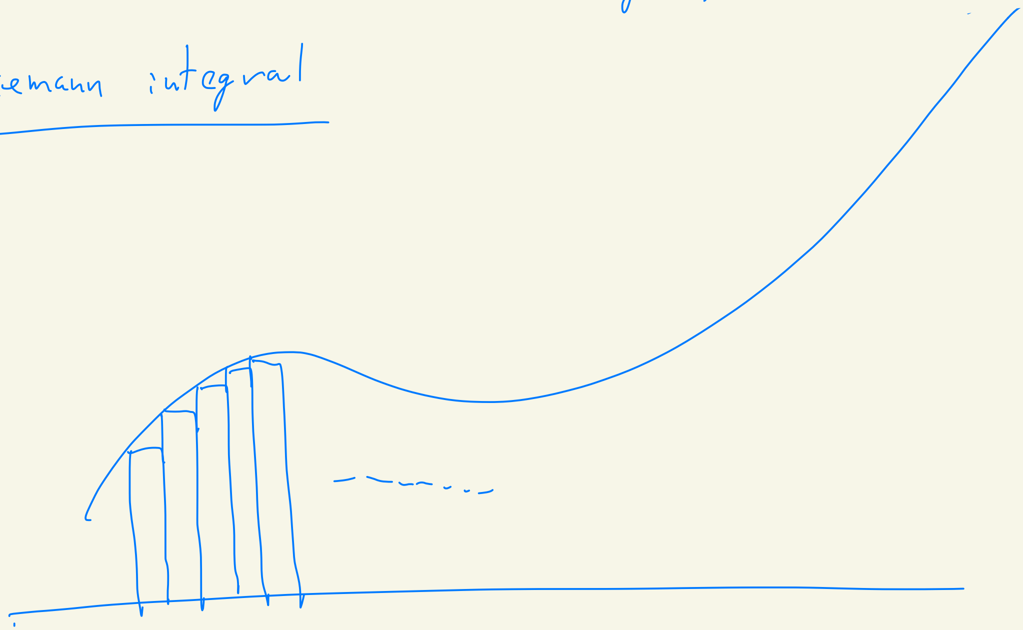
$$\mathbb{E}X \overset{\text{def}}{=} \underbrace{\lim_{n\to\infty} \sum_{i=-\infty}^{\infty} \frac{i}{n} \mathbb{P}\left(X \in \left[\frac{i}{n}, \frac{i+1}{n}\right)\right)}$$

this looks similar to

Riemann integral

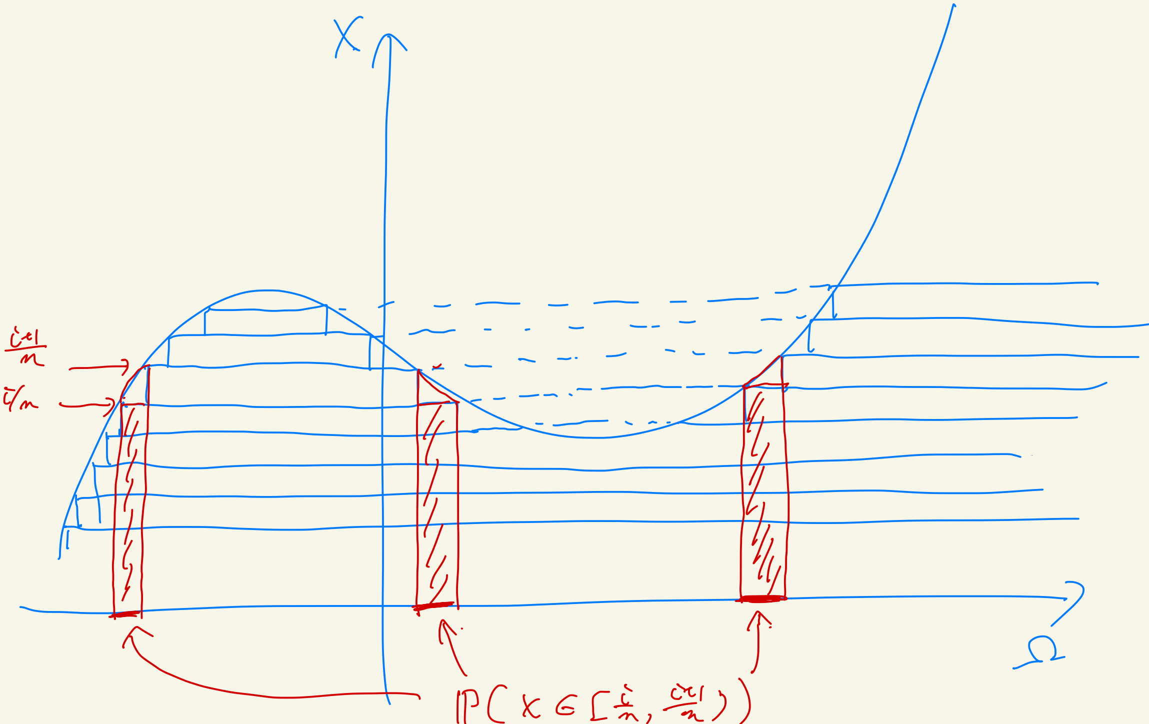
Difference between Riemann integral.

Riemann integral



Measure theory

$X : \Omega \to \mathbb{R}$



$$\mathbb{E}X \overset{\text{def}}{=} \lim_{n\to\infty} \sum_{i=-\infty}^{\infty} \frac{i}{n} \mathbb{P}\left(X \in \left[\frac{i}{n}, \frac{i+1}{n}\right]\right) = \int_\Omega X \, d\mathbb{P}$$

We can indeed show

$$\mathbb{E}X = \sum_{i=1}^{\infty} h_i \mathbb{P}(X = h_i) \quad \text{in discrete case}$$

$$\mathbb{E}X = \int_{-\infty}^{\infty} x\, p(x)\, dx \quad \text{in cont. case.}$$

Examples of σ-algebra

- Consider tossing a coin twice.

$$\Omega = \{HH, HT, TH, TT\}.$$

1) "Trivial" σ-algebra = Power set of Ω.

$$\mathcal{F} = P(\Omega)$$

$$\rightarrow |P(\Omega)| = 2^4 = 16$$

Note! For any Ω, $P(\Omega)$ is always a σ-algebra.

$\rightarrow$ However, $P(\Omega)$ is often useless for defining a measure.

2) non-trivial example

$$\text{Let } \mathcal{F} = \{\phi, \underbrace{\{HH\}, \{HT, TH, TT\}}_{\text{complement}}, \Omega\}$$

union makes

complement

$\rightarrow$ $\mathcal{F}$ is "generated" by $A = \{HH\}$

since this is the smallest σ-algebra containing $\{HH\}$.

Q. Is non-trivial σ-algebra always possible?

A. Yes, by "generating" σ-algebra from arbitrary collection of subsets.

Prop. Let $\mathcal{A}$ be a collection of subsets of Ω. There exists the smallest σ-algebra $\mathcal{F}$ containing $\mathcal{A}$, i.e. $\mathcal{A} \subset \mathcal{F}$.

Such $\mathcal{F}$ is denoted by $\sigma(\mathcal{A})$ and called σ-algebra generated by $\mathcal{A}$.

(pf) Recall that $P(\Omega)$ is a σ-algebra and $\mathcal{A} \subset P(\Omega)$.

Let $\mathcal{F} = \bigcap \mathcal{G}$ ← This definition is valid. $P(\Omega)$ is an example of $\mathcal{G}$

$\mathcal{G}$: all σ-algebra containing $\mathcal{A}$.

Since intersection of σ-algebras is also a σ-algebra, $\mathcal{F}$ defined above must be the smallest σ-algebra containing $\mathcal{A}$.

Q. How can we construct a "useful" σ-algebra on $\mathbb{R}$.

A. Yes.

We can generate a useful σ-algebra contain all intervals.

(a, b)
$[a, b]$
(a, b)
$[a, b]$

$\xrightarrow{\text{generate}}$

complement
countable union of these intervals
countable intersection

$\xrightarrow{\text{generate}}$

complement
countable union of above
countable intersection

$\xrightarrow{\text{repeat this process forever}}$

This σ-algebra is called Borel sets.

We denote this by $\mathcal{R}$.

Remark $\mathcal{R}$ contains

- all open sets
- all closed sets

) + combination of them

Def $A \in \mathcal{R} \iff A$ is "measurable"

Q. Is there any subset of $\mathbb{R}$ that is not measurable?

A. Yes, but the proof relies the Axiom of Choice.

Conditional probability

Original problem:

- What is the probability of some event A?
- $P(A)$ is determined by our probability measure.

New problem:

- Given that B happens, what is the probability of some event A?
- $P(A \mid B)$ is the conditional probability of the event A given B.

Conditional probability

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- What is the probability of some event A?
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New problem:

- Given that B happens, what is the probability of some event A?
- $P(A \mid B)$ is the conditional probability of the event A given B.

Example:

- Roll a die: $P(\{2\} \mid \text{even number})$

Conditional probability

(Definition)

Bayes' rule

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

Remark: Does conditional probability $P(\cdot \mid B)$ satisfy the axioms of a probability measure?

$$- \ \mathbb{P}(\phi \mid B) = \frac{\mathbb{P}(\phi \cap B)}{\mathbb{P}(B)} = \frac{0}{\mathbb{P}(B)} = 0$$

can apply countable additivity

$$- \ \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i \mid B\right) = \frac{\mathbb{P}\left(\left(\bigcup_{i=1}^{\infty} A_i\right) \cap B\right)}{\mathbb{P}(B)} = \frac{\boxed{\mathbb{P}\left(\bigcup_{i=1}^{\infty} (A_i \cap B)\right)}}{\mathbb{P}(B)}$$

A_i's are disjoint

$$= \frac{\sum_{i=1}^{\infty} \mathbb{P}(A_i \cap B)}{\mathbb{P}(B)}$$

$$= \sum_{i=1}^{\infty} \frac{\mathbb{P}(A_i \cap B)}{\mathbb{P}(B)} = \sum_{i=1}^{\infty} \mathbb{P}(A_i \mid B)$$

$$- \quad \mathbb{P}(\Omega \mid B) = \frac{\mathbb{P}(\Omega \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1 .$$

Conditional probability

Multiplication rule

$$P(A \cap B) = \boxed{P(A \mid B)P(B) = P(B \mid A)P(A)}$$

Bayes' rule $\mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(B \mid A)\,\mathbb{P}(A)}{\mathbb{P}(B)}$

Generalization:

Law of total probability

Let $A_1, A_2, \cdots, A_n$ be a partition of ~~a~~ Ω., such that $P(A_i) > 0$, then

$$P(B) = \sum_{i=1}^{n} P(A_i)P(B \mid A_i)$$

$$\mathbb{P}(B) = \sum_{i=1}^{n} \underline{\mathbb{P}(A_i \cap B)}$$

↳ rewrite this using the multiplication rule

Problem Set

Problem 1: Prove that for a σ-field $\mathcal{F}$, if $A_1, A_2, \dots \in \mathcal{F}$, then $\cap_{i=1}^{\infty} A_i \in \mathcal{F}$.

Problem 2: Prove monotonicity and subadditivity of measure μ on σ-field.

Problem 3: (Monty Hall problem) Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?
(Assumptions: the host will not open the door we picked and the host will only open the door which has a goat.)