UNIVERSITY OF TORONTO | Statistical Sciences

DoSS Summer Bootcamp Probability Module 7

Ichiro Hashimoto

University of Toronto

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Recap

Learnt in last module:

- Covariance
  - Covariance as an inner product
  - Correlation
  - Cauchy-Schwarz inequality
  - Uncorrelatedness and Independence
- Concentration
  - Markov's inequality
  - Chebyshev's inequality
  - Chernoff bounds

Outline

- Stochastic convergence
  - Convergence in distribution
  - Convergence in probability
  - Convergence almost surely
  - Convergence in L^p
  - Relationship between convergences

Stochastic Convergence

Recall: Convergence

Convergence of a sequence of numbers

A sequence $a_1, a_2, \cdots$ converges to a limit a if

$$\lim_{n\to\infty} a_n = a.$$

That is, for any $\epsilon > 0$, there exists an $N(\epsilon)$ such that

$$|a_n - a| < \epsilon, \quad \forall n > N(\epsilon).$$

Stochastic Convergence

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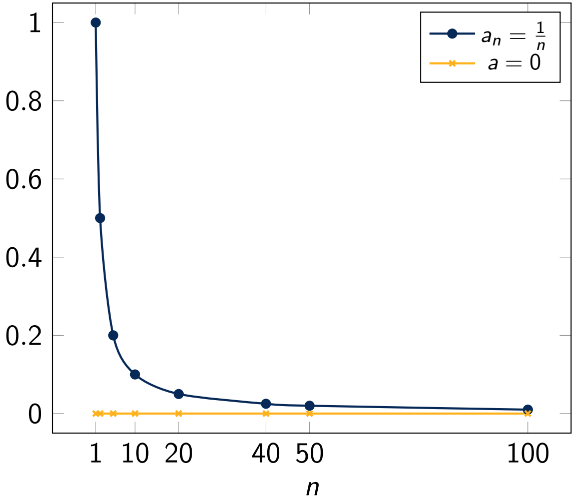
Example: $a_n = \frac{1}{n}$, $\forall \epsilon > 0$, take $N(\epsilon) = \lceil \frac{1}{\epsilon} \rceil$, then for $n > N(\epsilon)$,

$$|a_n - 0| = a_n < \epsilon, \quad \lim_{n\to\infty} a_n = 0.$$

Stochastic Convergence



- Capture the property of a series as $n \to \infty$;
- The limit is something where the series concentrate for large n;
- $|a_n - a|$ quantifies the closeness of the series and the limit.

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Stochastic Convergence

Observation: closeness of random variables

Sample mean of i.i.d. random variables

For i.i.d. random variables $X_i, i = 1, \cdots, n$ with $\mathbb{E}(X_i) = \mu$, $Var(X_i) = \sigma^2$, then for the sample mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$,

$$\mathbb{E}(\bar{X}) = \mu, \quad Var(\bar{X}) = \frac{\sigma^2}{n}.$$

Proof:

Stochastic Convergence

Example:

Further suppose $X_i, i = 1, \cdots, n$ i.i.d. with distribution $\mathcal{N}(\mu, \sigma^2)$, then $\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$, so we can draw the probability density plot of $\bar{X}$.

Stochastic Convergence

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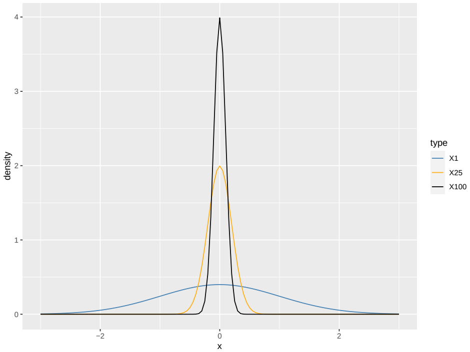

Figure: Probability density curve of sample mean of normal distribution

Stochastic Convergence

Intuition:

- Series of numbers $a_n \quad \Rightarrow \quad$ Series of random variables X_n;
- Limit $a \quad \Rightarrow \quad$ Limit X;
- How to quantify the closeness? $(|X_n - X|?)$

Stochastic Convergence

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Pointwise convergence / Sure convergence

Suppose random variables X_n and X are defined over the same probability space, then we say X_n converges to X pointwise if

$$\lim_{n \to \infty} X_n(\omega) = X(\omega), \ \forall \omega \in \Omega.$$

Stochastic Convergence

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Remark:

Incorporate probability measure in some sense.

Stochastic Convergence

Alternatives of describing the closeness:

- Utilize CDF: $F_{X_n}(x) - F_X(x)$;
- Utilize probability of an event: $\mathbb{P}(|X_n - X| > \epsilon)$;
- Utilize the probability over all ω: $\mathbb{P}(\lim_{n\to\infty} X_n(\omega) = X(\omega))$;
- Utilize mean/moments: $\mathbb{E}|X_n - X|^p$.

Stochastic Convergence

Convergence in distribution

A sequence $X_1, X_2, \cdots$ of real-valued random variables is said to converge in distribution, or converge weakly to a random variable X if

$$\lim_{n \to \infty} F_n(x) = F(x),$$

for every number $x \in \mathbb{R}$ at which $F(\cdot)$ is continuous. Here, $F_n(\cdot)$ and $F(\cdot)$ are the cumulative distribution functions of the random variables X_n and X, respectively.

Notation:

$X_n \xrightarrow{d} X, \quad X_n \xrightarrow{\mathcal{D}} X, \quad X_n \Rightarrow X.$

Stochastic Convergence

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Notation:

$X_n \xrightarrow{d} X, \quad X_n \xrightarrow{\mathcal{D}} X, \quad X_n \Rightarrow X.$

Remark:

X_n and X do not need to be defined on the same probability space.

Stochastic Convergence

Example:

Let $X_n = Z + \frac{1}{n}$, where $Z \sim \mathcal{N}(0, 1)$, then

- $X_n \xrightarrow{d} Z$,
- $X_n \xrightarrow{d} -Z$,
- $X_n \xrightarrow{d} Y$, $Y \sim \mathcal{N}(0, 1)$.

Proof:

Stochastic Convergence

Convergence in probability

A sequence X_n of random variables converges in probability towards the random variable X if for all $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\big(|X_n - X| > \epsilon\big) = 0.$$

Notation: $X_n \xrightarrow{p} X, \quad X_n \xrightarrow{P} X.$

Remark:
X_n and X need to be defined on the same probability space.

Stochastic Convergence

Examples:

- Let $X_n = Z + \frac{1}{n}$, where $Z \sim \mathcal{N}(0,1)$, then $X_n \xrightarrow{P} Z$.

  Proof:

- Let $X_n = Z + Y_n$, where $Z \sim \mathcal{N}(0,1)$, $\mathbb{E}(|Y_n|) = \frac{1}{n}$, then $X_n \xrightarrow{P} Z$.

  Proof:

Stochastic convergence

Convergence almost surely

A sequence X_n of random variables converges almost surely or almost everywhere or with probability 1 or strongly towards X means that

$$\mathbb{P}\left(\lim_{n\to\infty} X_n = X\right) = \mathbb{P}\left(\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right) = 1.$$

Notation: $X_n \xrightarrow{a.s.} X$.

Remark:
X_n and X need to be defined on the same probability space.

Stochastic convergence

Examples:

- Let $X_n = Z + \frac{1}{n}$, where $Z \sim \mathcal{N}(0, 1)$, then $X_n \xrightarrow{a.s.} Z$.

  Proof:

- Let $X_n = Z + Y_n$, where $Z \sim \mathcal{N}(0, 1)$, $\mathbb{E}(|Y_n|) = \frac{1}{n}$, do we have $X_n \xrightarrow{a.s.} Z$?

  Proof:

Stochastic convergence

Convergence in L^p

A sequence $\{X_n\}$ of random variables converges in L_p to a random variable X, $p \geq 1$, if

$$\lim_{n \to \infty} \mathbb{E}|X_n - X|^p = 0$$

Notation: $X_n \xrightarrow{L^p} X$.

Remark:

X_n and X need to be defined on the same probability space.

Stochastic convergence

Examples:

- Let $X_n = Z + \frac{1}{n}$, where $Z \sim \mathcal{N}(0,1)$, then $X_n \xrightarrow{L^p} Z$.

  Proof:

- Let $X_n = Z + Y_n$, where $Z \sim \mathcal{N}(0,1)$, $\mathbb{E}(|Y_n|^p) = \frac{1}{n}$, then $X_n \xrightarrow{L^p} Z$.

  Proof:

Stochastic convergence

Recall: A random variable $X \in L^p$ if $\|X\|_{L^p} = (E|X|^p)^{1/p} < \infty$.

$X_n \to X$ in L^p if $\lim_{n\to\infty} \|X_n - X\|_{L^p} = 0$

Monotonicity of L^p Convergence

If $q > p > 0$, L^q convergence implies L^p convergence

Proof:

Stochastic convergence

Recall: X_n converges to X in probability if for any $\epsilon > 0$ $\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0$.

L^p convergence implies Convergence in Probability

If $X_n \to X$ in L^p, then $X_n \to X$ in probability.

Proof:

Stochastic convergence

Recall: X_n converges to X in probability if for any $\epsilon > 0$ $\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0$.

a.s. Convergence implies Convergence in Probability

If $X_n \to X$ almost surely, then $X_n \to X$ in probability.

Proof:

Stochastic convergence

Recall: X_n converges to X in distribution if for any continuity point x of $P(X \leq x)$, $\lim_{n\to\infty} P(X_n \leq x) = P(X \leq x)$ holds.

Convergence in Probability implies Convergence in Distribution

If $X_n \to X$ in probability, then $X_n \to X$ in distribution.

Proof: Omitted

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Stochastic convergence

Relationship between convergences (on complete probability space):

$$\xrightarrow{L^s} \underset{s>r\geq 1}{\Rightarrow} \xrightarrow{L^r}$$

$$\Downarrow$$

$$\xrightarrow{\text{a.s.}} \Rightarrow \xrightarrow{p} \Rightarrow \xrightarrow{d}$$

Figure: relationship between convergences

Stochastic convergence

Highlights:

- Almost sure convergence implies convergence in probability:

$$X_n \xrightarrow{\text{a.s.}} X \quad \Rightarrow \quad X_n \xrightarrow{P} X;$$

- Convergence in probability implies convergence in distribution:

$$X_n \xrightarrow{P} X \quad \Rightarrow \quad X_n \xrightarrow{d} X;$$

- If X_n converges in distribution to a constant c, then X_n converges in probability to c:

$$X_n \xrightarrow{d} c \quad \Rightarrow \quad X_n \xrightarrow{P} c, \quad \text{provided } c \text{ is a constant.}$$

Problem Set

Problem 1: Prove that on a complete probability space, if $X_n \xrightarrow{L^p} X$, then $X_n \xrightarrow{P} X$. (Hint: use Markov's inequality)

Problem 2: Let $X_1, \cdots, X_n$ be i.i.d. random variables with $Bernoulli(p)$ distribution, and $X \sim Bernoulli(p)$ is defined on the same probability space, independent with X_i's. Does X_n converge in probability to X?

Problem 3: Give an example where X_n converges in distribution to X, but not in probability.