UNIVERSITY OF TORONTO | Statistical Sciences

DoSS Summer Bootcamp Probability Module 9

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Outline

- Counterexamples

Counterexamples

Recall: A random variable $X \in L^p$ if $\|X\|_{L^p} = (E|X|^p)^{1/p} < \infty$.

$X_n \to X$ in L^p if $\lim_{n\to\infty} \|X_n - X\|_{L^p} = 0$

Monotonicity of L^p Convergence

If $q > p > 0$, L^q convergence implies L^p convergence

Counterexample to the Converse:

Strategy Find X_n s.t. $X_n \to 0$ in L^p but $\mathbb{E}|X_n|^q \geq 1$

$$X_n = \begin{cases} n^{\alpha} & \text{w.p. } n^{-1} \\ 0 & \text{w.p. } 1 - n^{-1} \end{cases}$$

Find α later

$E|X_n|^p = n^{\alpha p - 1}$, $\mathbb{E}|X_n|^q = n^{\alpha q - 1}$

Let α s.t. $\alpha p - 1 < 0 \leq \alpha q - 1$

$\iff \underline{q^{-1} \leq \alpha < p^{-1}}$ For any α satisfying this $X_n \to 0$ in L^p while $\mathbb{E}|X_n|^q \geq 1$

Counterexamples

Recall: X_n converges to X in probability if for any $\epsilon > 0$ $\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0$.

L^p convergence implies Convergence in Probability

If $X_n \to X$ in L^p, then $X_n \to X$ in probability.

Counterexample to the Converse:

Strategy Find X_n s.t. $\mathbb{E}|X_n|^p \to \infty$ and $X_n \xrightarrow{P} 0$

$$X_n = \begin{cases} n^{1+\frac{1}{p}} & \text{w.p } \frac{1}{n} \\ 0 & \text{w.p } 1-\frac{1}{n} \end{cases}$$

Then for $\forall\, \varepsilon \in (0,1)$, $\mathbb{P}(|X_n| > \varepsilon) = \mathbb{P}(X_n = n^{1+\frac{1}{p}}) = \frac{1}{n} \to 0$.

Thus, $X_n \xrightarrow{P} 0$ holds.

However, $E|X_n|^p = n^{p+1} \cdot \frac{1}{n} = n^p \to \infty$.

Counterexamples

Recall: X_n converges to X in probability if for any $\epsilon > 0$ $\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0$.

a.s. Convergence implies Convergence in Probability

If $X_n \to X$ almost surely, then $X_n \to X$ in probability.

Counterexample to the Converse:

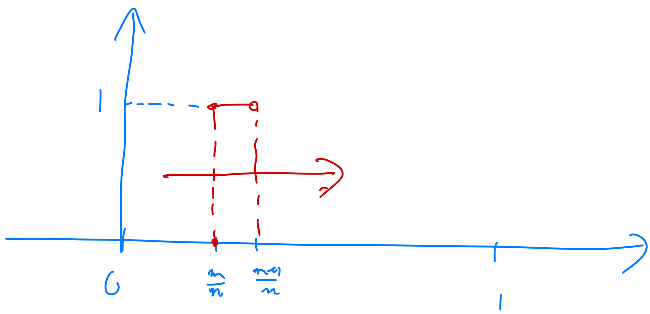
$\Omega = [0,1)$, $\mathbb{P}$ uniform on $[0,1)$

$$X_{n,m}(\omega) = \begin{cases} 1 & \text{if } \omega \in [\frac{m}{n}, \frac{m+1}{n}) \\ 0 & \text{else.} \end{cases}$$

$0 \leq m \leq n-1$

For $\varepsilon \in (0,1)$, $\mathbb{P}(|X_{n,m}| > \varepsilon) = \mathbb{P}(X_{n,m} = 1) = \frac{1}{n} \to 0$

For each n by taking $m = 0 \to n-1$



For fixed $\omega \in [0,1)$,

$X_{n,m}(\omega)$ takes both 0 and 1

while m runs from 0 to $n-1$ for each n.

Thus, $X_{n,m}(\omega)$ takes both 0 and 1 infinitely many times.

which means $X_{n,m}(\omega)$ does not converge to 0 everywhere.

This example is also counterexample to the following claim:

Claim: If $X_n \to X$ in L^p for any p,
then $X_n \to X$ a.s.

This is since $\mathbb{E} X_{n,m}^p = 1^p \cdot \frac{1}{n} = \frac{1}{n} \to 0$.

while $X_{n,m} \not\to 0$ a.s.

Counterexamples

Recall: X_n converges to X in distribution if for any continuity point x of $P(X \leq x)$, $\lim_{n\to\infty} P(X_n \leq x) = P(X \leq x)$ holds.

Convergence in Probability implies Convergence in Distribution

If $X_n \to X$ in probability, then $X_n \to X$ in distribution.

Counterexample to the Converse:

$\mathbb{P}(X=1) = \mathbb{P}(X=-1) = \frac{1}{2}.$

Let $X_n = (-1)^n X$

$X_n =_d X$ since distribution of X is symmetric.

Thus, $X_n \xrightarrow{d} X$ holds.

Let $\varepsilon \in (0, 2)$. Then for any odd n,

we have $\mathbb{P}(X_n \neq X) = 1$

Furthermore $\{X_n \neq X\} = \{|X_n - X| = 2\}$.

since $\varepsilon < 2$

Therefore,

$$1 = \mathbb{P}(X_n \neq X) = \mathbb{P}(|X_n - X| = 2) \leq \mathbb{P}(|X_n - X| > \varepsilon)$$

Thus, $\mathbb{P}(|X_n - X| > \varepsilon) = 1 \not\to 0$.

That means $X_n \not\to X$ in probability.

Counterexamples

Recall: X_n converges to X in distribution if for any continuity point x of $P(X \le x)$, $\lim_{n\to\infty} P(X_n \le x) = P(X \le x)$ holds.

Convergence in Probability implies Convergence in Distribution

If $X_n \to X$ in probability, then $X_n \to X$ in distribution.

Special case when the Converse holds:

If $X_n \xrightarrow{d} c$, then $X_n \xrightarrow{P} c$.

If X is constant, the converse holds.

Counterexamples

Monotone Convergence Theorem

If $X_n \geq 0$ and $X_n \nearrow X$, then $EX_n \nearrow EX$

$\geq c$, where c can be any constant

Counterexample when X_n is not lower bounded:

Strategy: Construct $X_n \nearrow 0$ while $E X_n = -\infty$.

(Counterexample 1.)

Let $\Omega = [0, c)$, where $c = \sum_{n=1}^{\infty} \frac{1}{n^2}$, $\mathbb{P} \sim$ uniform measure on $[0, c)$

$$X_n(\omega) = \begin{cases} 0 & \text{if } \omega \in \left[0, \sum_{k=1}^{n-1} \frac{1}{k^2}\right) \\ -m & \text{if } \omega \in \left[\sum_{k=1}^{m-1} \frac{1}{k^2}, \sum_{k=1}^{m} \frac{1}{k^2}\right), \ m \geq n \end{cases}$$

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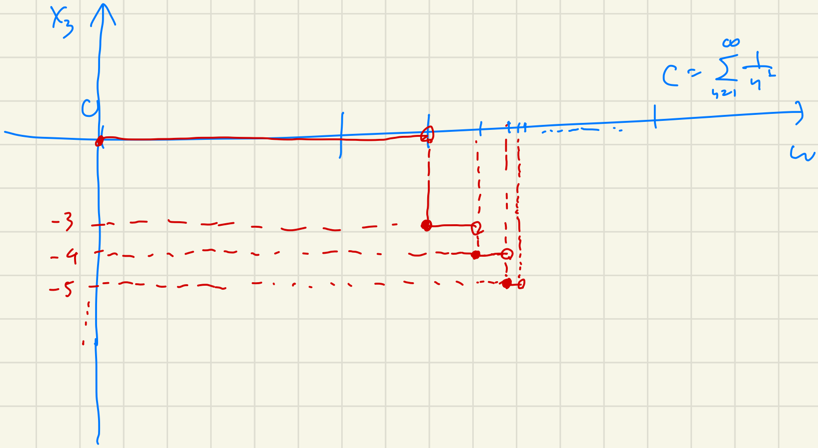


Then $X_n \nearrow 0$.

$$\mathbb{E}\, X_n = \sum_{m \ge n} (-m) \cdot \frac{1}{m^2}$$

$$= - \underbrace{\boxed{\sum_{m \ge n} \frac{1}{m}}}_{=\infty} = -\infty .$$

(Counterexample ②)

Let Y be $\mathbb{P}(Y = -2^i) = 2^{-i}$ for $i = 1, 2, \cdots$.

Then $\mathbb{P}(Y < 0) = 1$, i.e. $Y < 0$ a.s.

Let $X_n = \dfrac{Y}{n}$.

Since $Y < 0$ a.s. $\underline{X_n \nearrow 0}$ a.s.

However, $\underline{\mathbb{E}\, X_n} = \frac{1}{n} \mathbb{E}\, Y$

$$= \frac{1}{n} \sum_{i=1}^{\infty} (-2^i) \cdot 2^{-i} = -\frac{1}{n} \sum_{i=1}^{\infty} 1 = \underline{-\infty}$$

Counterexamples

Dominated Convergence Theorem

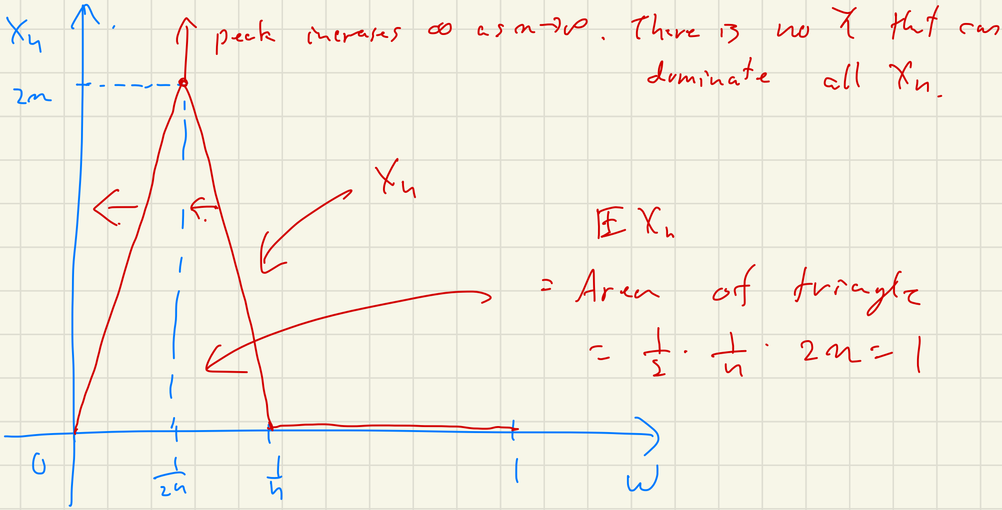
If $X_n \to X$ a.s. and $|X_n| \le Y$ a.s. for all n and Y is integrable, then $EX_n \to EX$

Counterexample when X_n is not dominated by an integrable random variable:

Strategy Find $X_n \to 0$ a.s. while $EX_n = 1 \not\to 0$.

Let $\Omega = (0,1)$, $\mathbb{P} \sim$ Unif on $(0,1)$

Let X_n defined as follows:



$\mathbb{E} X_n$

$= \text{Area of triangle}$

$= \frac{1}{2} \cdot \frac{1}{n} \cdot 2n = 1$

Triangle is shifting to left

$\Longrightarrow$ For fixed ω, $X_n(\omega) \longrightarrow 0$ as $n \to \infty$.

i.e. $X_n \to 0$ a.s.