Problem 1

Show that the probability density function of normal distribution $N(\mu, \sigma^2)$ integrates to 1.
(Hint: consider two normal random variables X, Y)

Solution:

Proof. Without loss of generality, let $\mu = 0, \sigma = 1$.
Consider two random variables $X, Y \sim \mathcal{N}(0, 1)$, then

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}), \quad f_Y(y) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{y^2}{2}).$$

Denote

$$A := \int_{-\infty}^{\infty} f_X(x)\, dx,$$

then our goal is to show $A = 1$.
Note that

$$\begin{aligned} A^2 &= \int_{-\infty}^{\infty} f_X(x)\, dx \cdot \int_{-\infty}^{\infty} f_Y(y)\, dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_X(x) f_Y(y)\, dxdy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp(-\frac{x^2 + y^2}{2})\, dxdy. \end{aligned}$$

Let $r = \sqrt{x^2 + y^2}$, $x = r \cos\theta$, then by change-of-variables in calculus, the Jacobian $|J| = r$, and the integral can be reformulated as

$$\begin{aligned} A^2 &= \int_0^{2\pi} \int_0^{\infty} \frac{1}{2\pi} \exp(-\frac{r^2}{2}) r\, drd\theta \\ &= (\int_0^{2\pi} d\theta) \cdot (\int_0^{\infty} \frac{1}{2\pi} \exp(-\frac{r^2}{2}) r\, dr) \\ &= 2\pi \cdot \frac{1}{2\pi} (-\exp(-\frac{r^2}{2}))|_0^{\infty} = 1. \end{aligned}$$

Therefore, we have $A = 1$ in view of $A \geq 0$. ■

Problem 2

Prove that for X with density function $f_X(x)$, the density function of $Y = X^2$ is

$$f_Y(y) = \frac{1}{2\sqrt{y}} (f_X(-\sqrt{y}) + f_X(\sqrt{y})), \quad y \geq 0.$$

(Hint: start by considering the CDF)

Solution:

Proof. Observe that

$$\mathbb{P}(X^2 \leq y) = \mathbb{P}(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}),$$

then by taking the derivative regarding y on both sides, the result follows. ■

Problem 3

Suppose $X_1, \cdots, X_n$ are i.i.d. sample following Uniform$[0,1]$ distribution, find the joint probability density function of $(X_{(1)}, X_{(n)})$.
(Hint: start by considering the CDF)

Solution:

Proof.

$$\begin{aligned}\mathbb{P}(X_{(1)} \leq x, X_{(n)} \leq y) &= \mathbb{P}(X_{(n)} \leq y) - \mathbb{P}(X_{(1)} > x, X_{(n)} \leq y) \\ &= \prod_{i=1}^{n} \mathbb{P}(X_i \leq y) - \prod_{i=1}^{n} \mathbb{P}(x < X_i \leq y) \\ &= (F_X(y))^n - (F_X(y) - F_X(x))^n.\end{aligned}$$

Taking the derivative regarding x, y on both sides, we have

$$f_{(X_{(1)}, X_{(n)})}(x, y) = n(n-1)(F_X(y) - F_X(x))^{n-2} f_X(x) f_X(y).$$

Further plugging in the CDF and PDF of uniform distribution, we have

$$f_{(X_{(1)}, X_{(n)})}(x, y) = n(n-1)(y-x)^{n-2}, \quad 0 \leq x < y \leq 1.$$

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