Problem 1

Prove that $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$ when X and Y are independent.
(Hint: simply consider the continuous case, use the independent property of the joint pdf)

Solution:

Proof. Assume both X, Y are continuous random variables and denote the joint PDF as $f_{(X,Y)}(x,y)$, then by independence, $f_{(X,Y)}(x,y) = f_X(x) \cdot f_Y(y)$

$$\begin{aligned}
\mathbb{E}(XY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{(X,Y)}(x,y)\ dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_X(x) \cdot f_Y(y)\ dxdy \\
&= (\int_{-\infty}^{\infty} xf_X(x)\ dx)(\int_{-\infty}^{\infty} yf_Y(y)\ dy) \\
&= \mathbb{E}(X)\mathbb{E}(Y).
\end{aligned}$$

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Problem 2

For $X \sim Uniform(a,b)$, compute $\mathbb{E}(X)$ and $\text{Var}(X)$.

Solution:

Proof. We have $f_X(x) = \frac{1}{b-a}, \quad a < x < b.$

$$\mathbb{E}(X) = \int_a^b x \cdot \frac{1}{b-a}\ dx = \frac{b+a}{2},$$

and

$$\mathbb{E}(X^2) = \int_a^b x^2 \cdot \frac{1}{b-a}\ dx = \frac{b^3 - a^3}{3(b-a)} = \frac{b^2 + ab + a^2}{3},$$

therefore,

$$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{(b-a)^2}{12}.$$

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Problem 3

Determine the MGF of $X \sim \mathcal{N}(\mu, \sigma^2)$.
(Hint: Start by considering the MGF of $Z \sim \mathcal{N}(0,1)$, and then use the transformation $X = \mu + \sigma Z$)

Solution:

Proof. For $Z \sim \mathcal{N}(0,1)$,

$$\begin{aligned} M_Z(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(tx - \frac{x^2}{2}\right) \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-t)^2}{2} + \frac{t^2}{2}\right) \mathrm{d}x \\ &= e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \mathcal{N}(t,1)\ \mathrm{d}x \\ &= e^{\frac{t^2}{2}}. \end{aligned}$$

Then for $X = \mu + \sigma Z \sim \mathcal{N}(\mu, \sigma^2)$,

$$M_X(t) = \mathbb{E}(e^{t(\mu+\sigma Z)}) = e^{t\mu} \cdot M_Z(\sigma t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}.$$

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Problem 4

The citizens of Remuera withdraw money from a cash machine according to $X = 50, 100, 200$ with probability $0.3, 0.5, 0.2$, respectively. The number of customers per day has the distribution $N \sim Poisson(\lambda = 10)$. Let $T_N = X_1 + X2 + \cdots + X_N$ be the total amount of money withdrawn in a day, where each X_i has the probability above, and X_i's are independent of each other and of N.

- Find $\mathbb{E}(T_N)$,
- Find $Var(T_N)$.

Solution:

Proof. Conditional on $N = n$, we have $\mathbb{E}(T_N \mid N = n) = \sum_{i=1}^{n} \mathbb{E}(X_i) = 105n$, and $Var(T_N \mid N = n) = Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i) = 2725n$. Therefore, by law of total expectation and variance,

$$\mathbb{E}(T_N) = \mathbb{E}(\mathbb{E}(T_N \mid N)) = \mathbb{E}(105N) = 105\mathbb{E}(N) = 1050,$$

and

$$\begin{aligned} Var(T_N) &= Var(\mathbb{E}(T_N \mid N)) + \mathbb{E}(Var(T_N \mid N)) \\ &= Var(105N) + \mathbb{E}(2725N) \\ &= 105^2 Var(N) + 2725\mathbb{E}(N) \\ &= 137500. \end{aligned}$$

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