Problem 1

Prove that on a complete probability space, if $X_n \xrightarrow{L^p} X$, then $X_n \xrightarrow{P} X$.
(Hint: use Markov's inequality)

Solution:

Proof. Consider for $\forall \epsilon > 0$,

$$\mathbb{P}(|X_n - X| > \epsilon) = \mathbb{P}(|X_n - X|^r > \epsilon^r) \leq \frac{\mathbb{E}(|X_n - X|^r)}{\epsilon^r} \to 0, \quad n \to \infty.$$

■

Problem 2

Let $X_1, \cdots, X_n$ be i.i.d. random variables with $Bernoulli(p)$ distribution, and $X \sim Bernoulli(p)$ is defined on the same probability space, independent with X_i's. Does X_n converge in probability to X?

Solution:

Proof. No. Consider for $\forall \epsilon \in (0, 1)$,

$$\mathbb{P}(|X_n - X| > \epsilon) = \mathbb{P}(X_n = 1, X = 0) + \mathbb{P}(X_n = 0, X = 1) = 2p(1 - p),$$

which does not converge to 0. ■

Problem 3

Give an example where X_n converges in distribution to X, but not in probability.

Solution:

Proof. Omitted. ■