



UNIVERSITY OF
TORONTO

Statistical Sciences

DoSS Summer Bootcamp Probability Module 7

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Recap

Learnt in last module:

- Covariance
 - ▷ Covariance as an inner product
 - ▷ Correlation
 - ▷ Cauchy-Schwarz inequality
 - ▷ Uncorrelatedness and Independence
- Concentration
 - ▷ Markov's inequality
 - ▷ Chebyshev's inequality
 - ▷ Chernoff bounds

Outline

- Stochastic convergence
 - ▷ Convergence in distribution
 - ▷ Convergence in probability
 - ▷ Convergence almost surely
 - ▷ Convergence in L^p
 - ▷ Relationship between convergences

Stochastic Convergence

Recall: Convergence

Convergence of a sequence of numbers

A sequence $a_1, a_2, \dots$ converges to a limit a if

$$\lim_{n \rightarrow \infty} a_n = a.$$

That is, for any $\epsilon > 0$, there exists an $N(\epsilon)$ such that

$$|a_n - a| < \epsilon, \quad \forall n > N(\epsilon).$$

Stochastic Convergence

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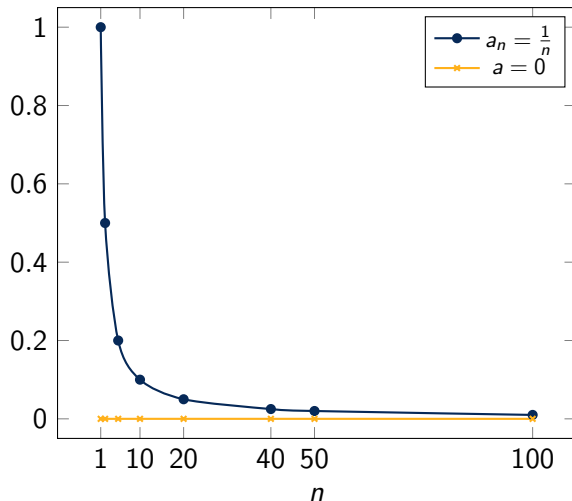
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$$|a_n - a| < \epsilon, \quad \forall n > N(\epsilon).$$

Example: $a_n = \frac{1}{n}$, $\forall \epsilon > 0$, take $N(\epsilon) = \lceil \frac{1}{\epsilon} \rceil$, then for $n > N(\epsilon)$,

$$|a_n - 0| = a_n < \epsilon, \quad \lim_{n \rightarrow \infty} a_n = 0.$$

Stochastic Convergence



- Capture the property of a series as $n \rightarrow \infty$;
- The limit is something where the series concentrate for large n ;
- $|a_n - a|$ quantifies the closeness of the series and the limit.

Stochastic Convergence

Observation: closeness of random variables

Sample mean of i.i.d. random variables

For i.i.d. random variables $X_i, i = 1, \dots, n$ with $\mathbb{E}(X_i) = \mu$, $\text{Var}(X_i) = \sigma^2$, then for the sample mean $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$,

$$\mathbb{E}(\bar{X}) = \mu, \quad \text{Var}(\bar{X}) = \frac{\sigma^2}{n}.$$

Proof:

Stochastic Convergence

Example:

Further suppose $X_i, i = 1, \dots, n$ i.i.d. with distribution $\mathcal{N}(\mu, \sigma^2)$, then $\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$, so we can draw the probability density plot of $\bar{X}$.

Stochastic Convergence

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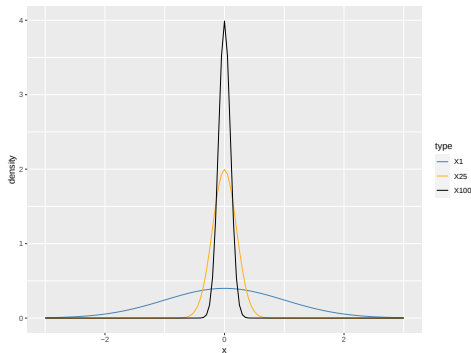


Figure: Probability density curve of sample mean of normal distribution

Stochastic Convergence

Intuition:

- Series of numbers $a_n \Rightarrow$ Series of random variables X_n ;
- Limit $a \Rightarrow$ Limit X ;
- How to quantify the closeness? ($|X_n - X|?$)

Stochastic Convergence

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Pointwise convergence / Sure convergence

Suppose random variables X_n and X are defined over the same probability space, then we say X_n converges to X pointwise if

$$\lim_{n \rightarrow \infty} X_n(\omega) = X(\omega), \quad \forall \omega \in \Omega.$$

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Remark:

Incorporate probability measure in some sense.

Stochastic Convergence

Alternatives of describing the closeness:

- Utilize CDF: $F_{X_n}(x) - F_X(x)$;
- Utilize probability of an event: $\mathbb{P}(|X_n - X| > \epsilon)$;
- Utilize the probability over all ω : $\mathbb{P}(\lim_{n \rightarrow \infty} X_n(\omega) = X(\omega))$;
- Utilize mean/moments: $\mathbb{E}|X_n - X|^p$.

Stochastic Convergence

Convergence in distribution

A sequence $X_1, X_2, \dots$ of real-valued random variables is said to converge in distribution, or converge weakly to a random variable X if

$$\lim_{n \rightarrow \infty} F_n(x) = F(x),$$

for every number $x \in \mathbb{R}$ at which $F(\cdot)$ is continuous. Here, $F_n(\cdot)$ and $F(\cdot)$ are the cumulative distribution functions of the random variables X_n and X , respectively.

Notation:

$$X_n \xrightarrow{d} X, \quad X_n \xrightarrow{\mathcal{D}} X, \quad X_n \Rightarrow X.$$

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Notation:

$$X_n \xrightarrow{d} X, \quad X_n \xrightarrow{\mathcal{D}} X, \quad X_n \Rightarrow X.$$

Remark:

X_n and X do not need to be defined on the same probability space.

Stochastic Convergence

Example:

Let $X_n = Z + \frac{1}{n}$, where $Z \sim \mathcal{N}(0, 1)$, then

- $X_n \xrightarrow{d} Z$,
- $X_n \xrightarrow{d} -Z$,
- $X_n \xrightarrow{d} Y$, $Y \sim \mathcal{N}(0, 1)$.

Proof:

Stochastic Convergence

Convergence in probability

A sequence X_n of random variables converges in probability towards the random variable X if for all $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}(|X_n - X| > \epsilon) = 0.$$

Notation: $X_n \xrightarrow{p} X$, $X_n \xrightarrow{P} X$.

Remark:

X_n and X need to be defined on the same probability space.

Stochastic Convergence

Examples:

- Let $X_n = Z + \frac{1}{n}$, where $Z \sim \mathcal{N}(0, 1)$, then $X_n \xrightarrow{P} Z$.

Proof:

- Let $X_n = Z + Y_n$, where $Z \sim \mathcal{N}(0, 1)$, $\mathbb{E}(|Y_n|) = \frac{1}{n}$, then $X_n \xrightarrow{P} Z$.

Proof:

Stochastic convergence

Convergence almost surely

A sequence X_n of random variables converges almost surely or almost everywhere or with probability 1 or strongly towards X means that

$$\mathbb{P} \left(\lim_{n \rightarrow \infty} X_n = X \right) = \mathbb{P} \left(\omega \in \Omega : \lim_{n \rightarrow \infty} X_n(\omega) = X(\omega) \right) = 1.$$

Notation: $X_n \xrightarrow{a.s.} X$.

Remark:

X_n and X need to be defined on the same probability space.

Stochastic convergence

Examples:

- Let $X_n = Z + \frac{1}{n}$, where $Z \sim \mathcal{N}(0, 1)$, then $X_n \xrightarrow{a.s.} Z$.

Proof:

- Let $X_n = Z + Y_n$, where $Z \sim \mathcal{N}(0, 1)$, $\mathbb{E}(|Y_n|) = \frac{1}{n}$, do we have $X_n \xrightarrow{a.s.} Z$?

Proof:

Stochastic convergence

Convergence in L^p

A sequence $\{X_n\}$ of random variables converges in L_p to a random variable X , $p \geq 1$, if

$$\lim_{n \rightarrow \infty} \mathbb{E}|X_n - X|^p = 0$$

Notation: $X_n \xrightarrow{L^p} X$.

Remark:

X_n and X need to be defined on the same probability space.

Stochastic convergence

Examples:

- Let $X_n = Z + \frac{1}{n}$, where $Z \sim \mathcal{N}(0, 1)$, then $X_n \xrightarrow{L^p} Z$.

Proof:

- Let $X_n = Z + Y_n$, where $Z \sim \mathcal{N}(0, 1)$, $\mathbb{E}(|Y_n|^p) = \frac{1}{n}$, then $X_n \xrightarrow{L^p} Z$.

Proof:

Stochastic convergence

Recall: A random variable $X \in L^p$ if $\|X\|_{L^p} = (E|X|^p)^{1/p} < \infty$.

$X_n \rightarrow X$ in L^p if $\lim_{n \rightarrow \infty} \|X_n - X\|_{L^p} = 0$

Monotonicity of L^p Convergence

If $q > p > 0$, L^q convergence implies L^p convergence

Proof:

Stochastic convergence

Recall: X_n converges to X in probability if for any $\epsilon > 0$ $\lim_{n \rightarrow \infty} P(|X_n - X| > \epsilon) = 0$.

L^p convergence implies Convergence in Probability

If $X_n \rightarrow X$ in L^p , then $X_n \rightarrow X$ in probability.

Proof:

Stochastic convergence

Recall: X_n converges to X in probability if for any $\epsilon > 0$ $\lim_{n \rightarrow \infty} P(|X_n - X| > \epsilon) = 0$.

a.s. Convergence implies Convergence in Probability

If $X_n \rightarrow X$ almost surely, then $X_n \rightarrow X$ in probability.

Proof:

Stochastic convergence

Recall: X_n converges to X in distribution if for any continuity point x of $P(X \leq x)$, $\lim_{n \rightarrow \infty} P(X_n \leq x) = P(X \leq x)$ holds.

Convergence in Probability implies Convergence in Distribution

If $X_n \rightarrow X$ in probability, then $X_n \rightarrow X$ in distribution.

Proof: Omitted

Stochastic convergence

Relationship between convergences (on complete probability space):

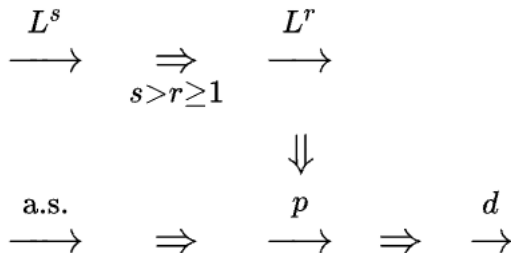


Figure: relationship between convergences

Stochastic convergence

Highlights:

- Almost sure convergence implies convergence in probability:

$$X_n \xrightarrow{\text{a.s.}} X \quad \Rightarrow \quad X_n \xrightarrow{P} X;$$

- Convergence in probability implies convergence in distribution:

$$X_n \xrightarrow{P} X \quad \Rightarrow \quad X_n \xrightarrow{d} X;$$

- If X_n converges in distribution to a constant c , then X_n converges in probability to c :

$$X_n \xrightarrow{d} c \quad \Rightarrow \quad X_n \xrightarrow{P} c, \quad \text{provided } c \text{ is a constant.}$$

Problem Set

Problem 1: Prove that on a complete probability space, if $X_n \xrightarrow{L^p} X$, then $X_n \xrightarrow{P} X$.
(Hint: use Markov's inequality)

Problem 2: Let $X_1, \dots, X_n$ be i.i.d. random variables with $Bernoulli(p)$ distribution, and $X \sim Bernoulli(p)$ is defined on the same probability space, independent with X_i 's. Does X_n converge in probability to X ?

Problem 3: Give an example where X_n converges in distribution to X , but not in probability.