



UNIVERSITY OF
TORONTO

Statistical Sciences

DoSS Summer Bootcamp Probability Module 9 - 10

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July 23-24, 2026

Counterexamples

Recall: A random variable $X \in L^p$ if $\|X\|_{L^p} = (E|X|^p)^{1/p} < \infty$.

$X_n \rightarrow X$ in L^p if $\lim_{n \rightarrow \infty} \|X_n - X\|_{L^p} = 0$

Monotonicity of L^p Convergence

If $q > p > 0$, L^q convergence implies L^p convergence

Counterexample to the Converse:

Counterexamples

Recall: X_n converges to X in probability if for any $\epsilon > 0$ $\lim_{n \rightarrow \infty} P(|X_n - X| > \epsilon) = 0$.

L^p convergence implies Convergence in Probability

If $X_n \rightarrow X$ in L^p , then $X_n \rightarrow X$ in probability.

Counterexample to the Converse:

Counterexamples

Recall: X_n converges to X in probability if for any $\epsilon > 0$ $\lim_{n \rightarrow \infty} P(|X_n - X| > \epsilon) = 0$.

a.s. Convergence implies Convergence in Probability

If $X_n \rightarrow X$ almost surely, then $X_n \rightarrow X$ in probability.

Counterexample to the Converse:

Counterexamples

Recall: X_n converges to X in distribution if for any continuity point x of $P(X \leq x)$, $\lim_{n \rightarrow \infty} P(X_n \leq x) = P(X \leq x)$ holds.

Convergence in Probability implies Convergence in Distribution

If $X_n \rightarrow X$ in probability, then $X_n \rightarrow X$ in distribution.

Counterexample to the Converse:

Counterexamples

Recall: X_n converges to X in distribution if for any continuity point x of $P(X \leq x)$, $\lim_{n \rightarrow \infty} P(X_n \leq x) = P(X \leq x)$ holds.

Convergence in Probability implies Convergence in Distribution

If $X_n \rightarrow X$ in probability, then $X_n \rightarrow X$ in distribution.

Special case when the Converse holds:

Counterexamples

a.s. convergence

Does a.s. convergence imply L^p convergence for any p ?

Counterexample for $p = 1$:

Counterexamples

a.s. convergence

Does L^p convergence imply a.s. convergence?

Counterexample:

Counterexamples

Monotone Convergence Theorem

If $X_n \geq 0$ and $X_n \nearrow X$, then $EX_n \nearrow EX$

Counterexample when X_n is not lower bounded:

Counterexamples

Dominated Convergence Theorem

If $X_n \rightarrow X$ a.s. and $|X_n| \leq Y$ a.s. for all n and Y is integrable, then $EX_n \rightarrow EX$

Counterexample when X_n is not dominated by an integrable random variable: