UNIVERSITY OF TORONTO | Statistical Sciences

DoSS Summer Bootcamp Probability Module 9 - 10

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Outline

- Counterexamples

Counterexamples

$X_n \to X$ in L^p $\iff$ $X_n - X \to 0$ in L^p

Recall: A random variable $X \in L^p$ if $\|X\|_{L^p} = (E|X|^p)^{1/p} < \infty$.

write = X_n

$X_n \to X$ in L^p if $\lim_{n\to\infty} \|X_n - X\|_{L^p} = 0$

$X = 0$

$\mathbb{E}|X_n|^p = \mathbb{E}|X_n - 0|^p$

Monotonicity of L^p Convergence

If $q > p > 0$, L^q convergence implies L^p convergence

Counterexample to the Converse:

Find X_n s.t. $X_n \to 0$ in L^p but $X_n \not\to 0$ in L^q.

$\mathbb{E}|X_n|^p \to 0$ but $\mathbb{E}|X_n|^q \not\to 0$

choose a so that the above is satisfied.

Let $\mathbb{P}(X_n = n^a) = \frac{1}{n}$, $\mathbb{P}(X_n = 0) = 1 - \frac{1}{n}$

$\mathbb{E}|X_n|^p = (n^a)^p \cdot \frac{1}{n} = n^{ap-1}$, $\mathbb{E}|X_n|^q = n^{aq-1}$

To make $\mathbb{E}|X_n|^p \to 0$, $ap - 1 < 0$ $(a < p^{-1})$

To make $\mathbb{E}|X_n|^q \not\to 0$, $aq - 1 \geq 0$. $(a \geq q^{-1})$

Thus by choosing $q^{-1} \leq a < p^{-1}$, X_n gives a counterexample.

Counterexamples

Recall: X_n converges to X in probability if for any $\epsilon > 0$ $\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0$.

L^p convergence implies Convergence in Probability

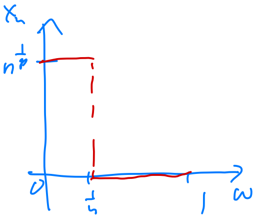
If $X_n \to X$ in L^p, then $X_n \to X$ in probability.

Counterexample to the Converse:

Need to find $X_n \xrightarrow{P} 0$ but $X_n \not\to 0$ in L^p.

Let $\Omega = [0,1]$

Define X_n s.t. $X_n(\omega) = \begin{cases} n^{\frac{1}{p}} & \text{if } \omega \in [0, \frac{1}{n}] \\ 0 & \text{otherwise} \end{cases}$



Then $\mathbb{E}|X_n|^p = (n^{\frac{1}{p}})^p \cdot \frac{1}{n} = 1 \not\to 0$. $X_n \not\to 0$ in L^p

We can actually show $X_n \to 0$ a.s.

For any $\omega \in (0,1]$, $X_n(\omega) = 0$ for $n > \omega^{-1}$ $(\Leftrightarrow \omega > n^{-1})$ $\therefore$ $\lim_{n\to\infty} X_n(\omega) = 0$ a.s.

Counterexamples

In module 7, we already discussed one counterexample.

Recall: X_n converges to X in probability if for any $\epsilon > 0$ $\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0$.

a.s. Convergence implies Convergence in Probability

If $X_n \to X$ almost surely, then $X_n \to X$ in probability.

Counterexample to the Converse:

We construct a counterexample using Borel-Cantelli lemma:

If A_n are independent and $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$,

then $\mathbb{P}(A_n \text{ occurring infinitely many times}) = 1$

Let X_n be independent r.v.s satisfying $\mathbb{P}(X_n = 1) = \frac{1}{n}$, $\mathbb{P}(X_n = 0) = 1 - \frac{1}{n}$

Then, for any $\varepsilon > 0$, $\mathbb{P}(|X_n| > \varepsilon) \leq \mathbb{P}(X_n = 1) = \frac{1}{n} \to 0$ $\therefore X_n \xrightarrow{P} 0$

Since X_n is independent and $\sum_{n=1}^{\infty} \mathbb{P}(X_n = 1) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$, by Borel-Cantelli lemma,

we have $\mathbb{P}(X_n = 1 \text{ for infinitely many } n) = 1$. That means $X_n \not\to 0$ a.s.

Counterexamples

Recall: X_n converges to X in distribution if for any continuity point x of $P(X \leq x)$, $\lim_{n\to\infty} P(X_n \leq x) = P(X \leq x)$ holds.

Convergence in Probability implies Convergence in Distribution

If $X_n \to X$ in probability, then $X_n \to X$ in distribution.

Counterexample to the Converse:

① Let $X_n = -X$ when $X \sim N(0,1)$.
Then $X_n = -X \sim N(0,1)$. Thus $X_n \xrightarrow{d} X$.

However $X_n = -X \not\to X$ in P.

② Let $\Omega = \{H, T\}$, with $\mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2}$.
Let $X_n(H) = 1$, $X_n(T) = 0$ and $X(T) = 1$, $X(H) = 0$
Then $\mathbb{P}(X_n = 1) = \frac{1}{2} = \mathbb{P}(X = 1)$ and $\mathbb{P}(X_n = 0) = \frac{1}{2} = \mathbb{P}(X = 0) \Rightarrow X_n \xrightarrow{d} X$
However $|X_n - X| = 1$ for any $n \rightarrow$ $X_n \xrightarrow{P} X$ fails

Both ① and ② are based on the same idea: choose X_n and X so that their distribution are identical

Counterexamples

Recall: X_n converges to X in distribution if for any continuity point x of $P(X \leq x)$, $\lim_{n\to\infty} P(X_n \leq x) = P(X \leq x)$ holds.

Convergence in Probability implies Convergence in Distribution

If $X_n \to X$ in probability, then $X_n \to X$ in distribution.

Special case when the Converse holds:

If $X = c$ (constant), then the converse holds.

$$X_n \xrightarrow{p} c \iff X_n \xrightarrow{d} c$$

Counterexamples

a.s. convergence

Does a.s. convergence imply L^p convergence for any p?

Counterexample for $p = 1$:

Same example for the failure of " $\xrightarrow{P}$ implies $\xrightarrow{L^p}$ " works.

Counterexamples

a.s. convergence

Does L^p convergence imply a.s. convergence?

Counterexample:

$\Omega = [0,1]$

$$X_{m,n}(\omega) = \begin{cases} 1 & \text{if } \omega \in \left[\frac{m}{n}, \frac{m+1}{n}\right] \\ 0 & \text{otherwise.} \end{cases} \qquad (0 \leq m \leq n-1)$$

$X_{m,n}(\omega)$ keeps occillating between 0 and 1 for every $\omega \in [0,1]$.

a.s. convergence fails

$\mathbb{E}|X_{m,n}|^p = 1^p \cdot \frac{1}{n} = \frac{1}{n} \to 0$.

L^p convergence holds

Counterexamples

Monotone Convergence Theorem

If $X_n \geq 0$ and $X_n \nearrow X$, then $EX_n \nearrow EX$

Counterexample when X_n is not lower bounded:

Let's find X_n s.t. $\mathbb{E} X_n = -\infty$ but $X_n \nearrow 0$ a.s.

Choose any Z s.t. $Z \leq 0$ and $\mathbb{E} Z = -\infty$, e.g. take absolute value of Cauchy distribution and multiply -1

Let $X_n = \frac{1}{n} \cdot Z$

Then, $X_n \nearrow 0$ a.s.

while $\mathbb{E} X_n = \frac{1}{n} \mathbb{E} Z = -\infty$

Counterexamples

$\mathbb{E}|Y| < \infty$

Dominated Convergence Theorem

If $X_n \to X$ a.s. and $|X_n| \leq Y$ a.s. for all n and Y is integrable, then $EX_n \to EX$

Counterexample when X_n is not dominated by an integrable random variable:

Q1. Isn't $X_n \to X$ a.s enough?

A. The previous example is a counterexample.

Q2. How about assuming $\mathbb{E}|X_n| \leq M < \infty$ instead?

A. Let $\Omega = [0,1]$. $X_n : \Omega \to \mathbb{R}$ be.

$$X_n(\omega) = \begin{cases} n & \text{if } \omega \in [0, \frac{1}{n}] \\ 0 & \text{otherwise.} \end{cases}$$

Then $\lim_{n\to\infty} X_n(\omega) = 0$ for $\forall \omega \in [0,1]$. $\therefore X_n \to 0$ a.s.

$\mathbb{E}|X_n| = n \cdot \frac{1}{n} = 1 < \infty$

However, $\mathbb{E}X_n = 1 \not\to 0 = \mathbb{E}0$.